Unsteady interaction of a viscous fluid with an elastic shell modeled by full von Karman equations

Igor Chueshov* and Iryna Ryzhkova†

Department of Mechanics and Mathematics,
Kharkov National University,
Kharkov, 61077, Ukraine

December 30, 2011

Abstract

We study well-posedness and asymptotic dynamics of a coupled system consisting of linearized 3D Navier–Stokes equations in a bounded domain and a classical (nonlinear) full von Karman shallow shell equations that accounts for both transversal and lateral displacements on a flexible part of the boundary. We also take into account rotational inertia of filaments of the shell. Out main result shows that the problem generates a semiflow in an appropriate phase space. The regularity provided by viscous dissipation in the fluid allows us to consider simultaneously both cases of presence inertia in the lateral displacements and its absence. Our second result states the existence of a compact global attractor for this semiflow in the case of presence of (rotational) damping in the transversal component and a particular structure of external forces.

Keywords: Fluid–structure interaction, linearized 3D Navier–Stokes equations, nonlinear shell, global attractor.
2010 MSC: 74F10, 35B41, 35Q30, 74K20

1 Introduction

We consider a coupled (hybrid) system which describes an interaction of a homogeneous viscous incompressible fluid which occupies a domain $\mathcal{O}$ bounded by the (solid) walls of the container S and a horizontal (flat) boundary Ω on which a thin (nonlinear) elastic shell is placed. The motion of the fluid is described by linearized 3D Navier–Stokes equations. To describe deformations of the shell we use the full von Karman shallow shell model which accounts for both transversal and in-plane displacements. For details

*e-mail: chueshov@univer.kharkov.ua
†e-mail: iryonok@gmail.com

concerning the shell model chosen we refer to [42, 26, 27] and also to the papers [28, 29, 30, 33, 35, 36] and the references therein.

This fluid-structure interaction model assumes that large deflections of the shell produce small effect on the fluid. This corresponds to the case when the fluid fills the container which is large in comparison with the size of the plate.

We note that the mathematical studies of the problem of fluid–structure interaction in the case of viscous fluids and elastic plates/bodies have a long history. We refer to [8, 9, 13, 19, 20, 21, 24] and the references therein for the case of plates/membranes, to [15] in the case of moving elastic bodies, and to [1, 2, 3, 6, 7, 16] in the case of elastic bodies with the fixed interface; see also the literature cited in these papers.

We also note that the global (asymptotic) dynamics in nonlinear plate-fluid models were studied before in [9, 13]. The article [9] deals with a class of fluid-plate interaction problems, when the plate, occupying Ω, oscillates in *longitudinal* directions only. This kind of models arises in the study of blood flows in large arteries (see, e.g., [19] and the references therein). A fluid-plate interaction model, accounting for *only transversal* displacement of the plate, was studied in [13]. In contrast our mathematical model formulated below takes into account *both* transversal and in-plane displacements.

Let $\mathcal{O} \subset \mathbb{R}^3$ be a bounded domain with a sufficiently smooth boundary $\partial\mathcal{O}$. We assume that $\partial\mathcal{O} = \Omega \cup S$, where

$$\Omega \subset \{x = (x_1; x_2; 0) \,:\, x' \equiv (x_1; x_2) \in \mathbb{R}^2\}$$

with a smooth contour $\Gamma = \partial\Omega$ and S is a surface which lies in the subspace $\mathbb{R}^3_- = \{x_3 \le 0\}$. The exterior normal on $\partial\mathcal{O}$ is denoted by n. We have that $n = (0; 0; 1)$ on Ω.

We consider the following *linear* Navier–Stokes equations in $\mathcal{O}$ for the fluid velocity field $v = v(x,t) = (v^1(x,t); v^2(x,t); v^3(x,t))$ and for the pressure $p(x,t)$:

$$v_t - \nu\Delta v + \nabla p = G_f(t) \quad \text{in} \quad \mathcal{O} \times (0, +\infty), \tag{1}$$

$$\operatorname{div} v = 0 \quad \text{in} \quad \mathcal{O} \times (0, +\infty), \tag{2}$$

where $\nu > 0$ is the dynamical viscosity and $G_f(t)$ is a volume force (which may depend on t). We supplement (1) and (2) with the (non-slip) boundary conditions imposed on the velocity field $v = v(x,t)$:

$$v = 0 \ \text{ on } S; \quad v \equiv (v^1; v^2; v^3) = (u^1_t; u^2_t; w_t) \ \text{ on } \Omega, \tag{3}$$

where $u = u(x,t) \equiv (u^1; u^2; w)(x,t)$ is the displacement of the shell occupying Ω. Here w stands for transversal displacement, $\bar{u} = (u^1; u^2)$ — for lateral (in-plane) displacements.

To describe the shell motion we use the full von Karman model which takes into account the rotational inertia of the filaments and possible presence of in-plane acceleration terms (see the literature cited above).

We denote by $T_f(v)$ the surface force exerted by the fluid on the shell, which is equal to $Tn|_\Omega$, where n is a outer unit normal to $\partial\mathcal{O}$ at Ω and $T = \{T_{ij}\}_{i,j=1}^3$ is the stress tensor of the fluid,

$$T_{ij} \equiv T_{ij}(v) = \nu\left(v^i_{x_j} + v^j_{x_i}\right) - p\delta_{ij}, \quad i,j = 1,2,3.$$

Since $n = (0;0;1)$ on Ω, we have that

$$T_f(v) = (\nu(v^1_{x_3} + v^3_{x_1}); \nu(v^2_{x_3} + v^3_{x_2}); 2\nu\partial_{x_3}v^3 - p).$$

Below for some simplification we assume that for the case considered Young's modulus E and Poisson's ratio $\mu \in (0,1/2)$ are such that $Eh = 2(1+\mu)$, where h is the thickness of the shell. In this case the (elastic) stress tensor $\{N_{ij}\}$ is given by the formulas

$$N_{11} = \frac{2}{1-\mu}\left(\varepsilon_{11} + \mu\varepsilon_{22}\right), \quad N_{22} = \frac{2}{1-\mu}\left(\varepsilon_{22} + \mu\varepsilon_{11}\right), \quad N_{12} = \varepsilon_{12}, \tag{4}$$

where the deformation tensor $\{\varepsilon_{ij}\}$ has the form

$$\begin{aligned}
\varepsilon_{11} &= u^1_{x_1} + k_1 w + \frac{1}{2}(w_{x_1})^2, \\
\varepsilon_{22} &= u^2_{x_2} + k_2 w + \frac{1}{2}(w_{x_2})^2, \\
\varepsilon_{12} &= u^1_{x_2} + u^2_{x_1} + w_{x_1}w_{x_2}.
\end{aligned}$$

Here k_1 and k_2 are curvatures of the initial form of the shell which are sufficiently smooth functions of $x' \in \Omega$.

After an appropriate rescaling of the parameters and functions we can model shell dynamics by the following equations

$$\begin{aligned}
&M_\alpha(w_{tt} + \gamma w_t) + \Delta^2 w + k_1 N_{11} + k_2 N_{22} \\
&\qquad - \partial_{x_1}(N_{11}w_{x_1} + N_{12}w_{x_2}) - \partial_{x_2}(N_{12}w_{x_1} + N_{22}w_{x_2}) \\
&\qquad\qquad = G_3(t) - 2\nu\partial_{x_3}v^3 + p \ \text{ in } \ \Omega\times(0,\infty),
\end{aligned} \tag{5}$$

where $M_\alpha = 1 - \alpha\Delta$, and

$$\begin{aligned}
\varrho u^1_{tt} &= \partial_{x_1}N_{11} + \partial_{x_2}N_{12} + G_1(t) - \nu(v^1_{x_3} + v^3_{x_1}), \\
\varrho u^2_{tt} &= \partial_{x_1}N_{12} + \partial_{x_2}N_{22} + G_2(t) - \nu(v^2_{x_3} + v^3_{x_2}),
\end{aligned} \tag{6}$$

where $G_{sh}(t) \equiv (G_1; G_2; G_3)(t)$ is a given body force applied to the shell. $\alpha > 0$ and $\varrho \geq 0$ are constants which take into account rotational inertia and in-plane inertia of the shell, respectively, γ is a non-negative parameter which describes intensity of the viscous damping of the shell material.

We impose the clamped boundary conditions on the shell

$$u^1|_{\partial\Omega} = u^2|_{\partial\Omega} = w|_{\partial\Omega} = \left.\frac{\partial w}{\partial n}\right|_{\partial\Omega} = 0 \tag{7}$$

and supply (1)–(7) with initial data for the velocity field $v = (v^1; v^2; v^3)$ and the shell displacement vector $u = (u^1; u^2; w)$ of the form[1]

$$v\big|_{t=0} = v_0, \quad u\big|_{t=0} = u_0, \quad w_t\big|_{t=0} = w_1, \quad \varrho\left[\bar{u}_t\big|_{t=0} - \bar{u}_1\right] = 0, \qquad (8)$$

where $\bar{u} = (u^1; u^2)$. Here $v_0 = (v_0^1; v_0^2; v_0^3)$, $u_0 = (u_0^1; u_0^2; w_0)$, w_1, and $\bar{u}_1 = (u_1^1; u_1^2)$ are given vector functions which we specify later.

We note that (2) and (3) imply the following compatibility condition

$$\int_\Omega w_t(x', t)dx' = 0 \quad \text{for all} \quad t \geq 0. \qquad (9)$$

This condition fulfills when

$$\int_\Omega w(x', t)dx' = const \quad \text{for all} \quad t \geq 0$$

and can be interpreted as preservation of the volume of the fluid.

We emphasize that even in the linear case we cannot split system (1)–(8) into two sets of equations describing longitudinal and transversal plate movements separately, i.e., we cannot reduce the model considered to the cases studied in [9, 13]. The point is that the surface force $T_f(v)$ is not the sum of the corresponding loads in the models [9] and [13]. The models in [9, 13] are much simpler in several respects. For instance, in the case of longitudinal plate deformations only (see [9]) the equations which correspond to (6) do not contain the terms $v^3_{x_i}$ and the model does not require any compatibility conditions like (9) because the volume of the fluid obviously preserves in the case of longitudinal deformations. In the case of purely transversal displacements [13] the force exerted on the plate by the fluid contains the pressure only.

In the following remark we describe some structural properties of the shell model chosen.

Remark 1.1 **(A)** One can see that the equations for the in-plane displacement vector $\bar{u} = (u^1; u^2)$ can be written in the vector form as follows:

$$\varrho\bar{u}_{tt} + A\bar{u} = B(w) + (G_1(t) - \nu(v^1_{x_3} + v^3_{x_1}); G_2(t) - \nu(v^2_{x_3} + v^3_{x_2})), \qquad (10)$$

where the operator A in (10) is defined by

$$A = -\begin{pmatrix} (1+\lambda)\partial^2_{x_1} + \partial^2_{x_2} & \lambda\partial_{x_1x_2} \\ \lambda\partial_{x_1x_2} & \partial^2_{x_1} + (1+\lambda)\partial^2_{x_2} \end{pmatrix}$$

with $\lambda = \frac{1+\mu}{1-\mu}$ and $D(A) = [H^2(\Omega)\bigcap H^1_0(\Omega)]^2$. The nonlinear term $B(w)$ in (10) has the form

$$B(w) = \begin{pmatrix} \dfrac{1}{1-\mu}\partial_{x_1}\left[2\varkappa_1 w + (w_{x_1})^2 + \mu(w_{x_2})^2\right] + \partial_{x_2}\left[w_{x_1}w_{x_2}\right] \\ \partial_{x_1}\left[w_{x_1}w_{x_2}\right] + \dfrac{1}{1-\mu}\partial_{x_2}\left[2\varkappa_2 w + (w_{x_2})^2 + \mu(w_{x_1})^2\right] \end{pmatrix},$$

[1] We put the multiplier ϱ in the fourth relation of (8) to emphasize that this relation is not needed in the case of negligibly small in-plane inertia ($\varrho = 0$).

where $\varkappa_1 = k_1+\mu k_2$ and $\varkappa_2 = k_2+\mu k_1$. Thus for fixed w and v the equations for the in-plain displacement $\bar{u}$ are similar to the standard equations of $2D$ (linear) elasticity theory.

(B) The equations in (5) and (6) can be also written in the form

$$(1-\alpha\Delta)(w_{tt}+\gamma w_t)+\Delta^2 w+\mathrm{trace}\,\{K\mathscr{N}(u)\} \\ = \mathrm{div}\,\{\mathscr{N}(u)\nabla w\}+G_3(t)-2\nu\partial_{x_3}v^3+p$$

and

$$\varrho\bar{u}_{tt}=\mathrm{div}\,\{\mathscr{N}(u)\}+\begin{pmatrix} G_1(t)-\nu(v^1_{x_3}+v^3_{x_1}) \\ G_2(t)-\nu(v^2_{x_3}+v^3_{x_2}) \end{pmatrix},$$

where $K=\mathrm{diag}\,(k_1,k_2)$ and

$$\mathscr{N}(u)\equiv\begin{pmatrix} N_{11} & N_{12} \\ N_{12} & N_{22}\end{pmatrix}=\mathscr{C}(\epsilon_0(\bar{u})+wK+f(\nabla w))$$

with $\bar{u}=(u_1;u_2)$, $\mathscr{C}(\epsilon)=2(1-\mu)^{-1}\left[\mu\,\mathrm{trace}\,\epsilon\cdot I+(1-\mu)\epsilon\right]$ and

$$\epsilon_0(\bar{u})=\frac{1}{2}(\nabla\bar{u}+\nabla^T\bar{u}),\quad f(s)=\frac{1}{2}s\otimes s,\ s\in\mathbb{R}^2.$$

This form of the full von Karman system was used earlier by many authors in the case when the fluid velocity field v is absent and $k_i\equiv 0$ (see, e. g., [33] or [23] and the references therein).

Our main goal in this paper is to prove well-posedness of the problem in (1)–(9) in the class of finite energy solutions (see Theorem 3.3) and to show a possibility of compact long-time dynamics (see Theorem 5.1 on the existence of a compact global attractor). We note for the proof of uniqueness in Theorem 3.3 relies substantially on the H^1-regularity of w_t, which follows from the structure of the mass operator M_α and involves Sedenko's method (see [35, 36]). To prove the existence of a global attractor in Theorem 5.1 we use J.Ball's method (see [5] and [32]). To apply this method we need the property $\gamma>0$, i.e., assume a presence of rotational damping in the transversal component of displacement. The question whether the system under consideration demonstrates compact long-time behavior without mechanical damping in the shell component is still open. In contrast we note that the existence of global attractors in the models considered in [9, 13] *does not require* any mechanical damping and compact asymptotic dynamics of the corresponding system is guaranteed by viscous dissipation of fluid. One of the reasons for this is that the models in [9, 13] do not include higher order (rotational type) inertial terms and thus the finiteness of the *full* dissipation integral for the displacement follows from the viscosity of the fluid via the compatibility condition in (3).

The paper organized as follows. In Section 2 we provide some preliminary material related to Sobolev spaces and the Stokes problem. In Section 3 we state and prove our main well-posedness result. Section 5 deals with long-time dynamics.

2 Preliminaries

In this section we introduce Sobolev type spaces we need and provide with some results concerning the Stokes problem.

2.1 Spaces and notations

To introduce Sobolev spaces we follow approach presented in [40].

Let D be a sufficiently smooth domain and $s \in \mathbb{R}$. We denote by $H^s(D)$ the Sobolev space of order s on the set D which we define as a restriction (in the sense of distributions) of the space $H^s(\mathbb{R}^d)$ (introduced via Fourier transform). We define the norm in $H^s(D)$ by the relation

$$\|u\|_{s,D}^2 = \inf\left\{\|w\|_{s,\mathbb{R}^d}^2 \ :\ w \in H^s(\mathbb{R}^d), \quad w = u \ \text{ on } \ D\right\}$$

We also use the notation $\|\cdot\|_D = \|\cdot\|_{0,D}$ and $(\cdot,\cdot)_D$ for the corresponding L_2 norm and inner product. We denote by $H_0^s(D)$ the closure of $C_0^\infty(D)$ in $H^s(D)$ (with respect to $\|\cdot\|_{s,D}$) and introduce the spaces

$$H_*^s(D) := \left\{f\big|_D \ :\ f \in H^s(\mathbb{R}^d),\ \operatorname{supp} f \subset \overline{D}\right\}, \quad s \in \mathbb{R}.$$

Below we need them to describe boundary traces on $\Omega \subset \partial\mathcal{O}$. We endow the classes $H_*^s(D)$ with the induced norms $\|f\|_{s,D}^* = \|f\|_{s,\mathbb{R}^d}$ for $f \in H_*^s(D)$. It is clear that

$$\|f\|_{s,D} \le \|f\|_{s,D}^*, \quad f \in H_*^s(D).$$

However, in general the norms $\|\cdot\|_{s,D}$ and $\|\cdot\|_{s,D}^*$ are not equivalent. It is known that (see [40, Theorem 4.3.2/1]) that $C_0^\infty(D)$ is dense in $H_*^s(D)$ and

$$\begin{aligned}
&H_*^s(D) \subset H_0^s(D) \subset H^s(D), \quad s \in \mathbb{R};\\
&H_0^s(D) = H^s(D), \quad -\infty < s \le 1/2;\\
&H_*^s(D) = H_0^s(D), \quad -1/2 < s < \infty, \quad s - 1/2 \notin \{0,1,2,\dots\}.
\end{aligned}$$

In particular, $H_*^s(D) = H_0^s(D) = H^s(D)$ for $|s| < 1/2$. Note that in the notations of [31] the space $H_*^{m+1/2}(D)$ is the same as $H_{00}^{m+1/2}(D)$ for every $m = 0,1,2,\dots$, and for $s = m + \sigma$ with $0 < \sigma < 1$ we have

$$\|u\|_{s,D}^* = \left\{\|u\|_{s,D}^2 + \sum_{|\alpha|=m} \int_D \frac{|D^\alpha u(x)|^2}{d(x,\partial D)^{2\sigma}}\, dx\right\}^{1/2},$$

where $d(x,\partial D)$ is the distance between x and ∂D. The norm $\|\cdot\|_{s,D}^*$ is equivalent to $\|\cdot\|_{s,D}$ in the case when $s > -1/2$ and $s - 1/2 \notin \{0,1,2,\dots\}$.

Understanding adjoint spaces with respect to duality between $C_0^\infty(D)$ and $[C_0^\infty(D)]'$ by Theorems 4.8.1 and 4.8.2 from [40] we also have that

$$[H_*^s(D)]' = H^{-s}(D),\ s \in \mathbb{R}, \quad \text{and} \quad [H^s(D)]' = H_*^{-s}(D),\ s \in (-\infty, 1/2).$$

Below we also use the factor-spaces $H^s(D)/\mathbb{R}$ with the naturally induced norm.

To describe fluid velocity fields we introduce the following spaces.

Let $\mathscr{C}(\mathcal{O})$ be the class of C^∞ vector-valued solenoidal (i.e., divergence-free) functions on $\overline{\mathcal{O}}$ which vanish in a neighborhood of S. We denote by X the closure of $\mathscr{C}(\mathcal{O})$ with respect to the L_2-norm and by V the closure with respect to the $H^1(\mathcal{O})$-norm. One can see that

$$X = \left\{ v = (v^1; v^2; v^3) \in [L_2(\mathcal{O})]^3 \;:\; \operatorname{div} v = 0, \gamma_n v \equiv (v, n) = 0 \text{ on } S \right\}; \tag{11}$$

and

$$V = \left\{ v = (v^1; v^2; v^3) \in [H^1(\mathcal{O})]^3 \;:\; \operatorname{div} v = 0,\ v = 0 \text{ on } S \right\}.$$

We equip X with L_2-type norm $\|\cdot\|_{\mathcal{O}}$ and denote by $(\cdot,\cdot)_{\mathcal{O}}$ the corresponding inner product. The space V is endowed with the norm $\|\cdot\|_V = \|\nabla\cdot\|_{\mathcal{O}}$. For some details concerning this type spaces we refer to [39], for instance.

We also need the Sobolev spaces consisting of functions with zero average on the domain Ω, namely we consider the space

$$\widehat{L}_2(\Omega) = \left\{ u \in L_2(\Omega) : \int_\Omega u(x')dx' = 0 \right\}$$

and also $\widehat{H}^s(\Omega) = H^s(\Omega) \cap \widehat{L}_2(\Omega)$ for $s > 0$ with the standard $H^s(\Omega)$-norm. The notations $\widehat{H}^s_*(\Omega)$ and $\widehat{H}^s_0(\Omega)$ have a similar meaning.

To describe shell displacement we use the spaces

$$W = H^1_0(\Omega) \times H^1_0(\Omega) \times H^2_0(\Omega), \quad Y = L_2(\Omega) \times L_2(\Omega) \times H_\alpha, \tag{12}$$

where H_α equals to $\widehat{H}^1_0(\Omega)$ with the equivalent norm $||\cdot||^2_{H_\alpha} = ||\cdot||^2_\Omega + \alpha||\nabla\cdot||^2_\Omega$ and corresponding inner product.

Remark 2.1 Below we also use $\widehat{H}^2_0(\Omega)$ as a state space for the displacement of the plate. It is clear that $\widehat{H}^2_0(\Omega)$ is a closed subspace of $H^2_0(\Omega)$. We denote by $\widehat{P}$ the projection on $\widehat{H}^2_0(\Omega)$ in $H^2_0(\Omega)$ which is orthogonal with respect to the inner product $(\Delta\cdot, \Delta\cdot)_\Omega$. One can see that $(I - \widehat{P})H^2_0(\Omega)$ consists of functions $u \in H^2_0(\Omega)$ such that $\Delta^2 u = const$ and thus has dimension one.

2.2 Stokes problem

In further considerations we need some regularity properties of the terms responsible for fluid–plate interaction. To this end we consider the following Stokes problem

$$\begin{aligned} -\nu\Delta v + \nabla p = g, \quad \operatorname{div} v = 0 \quad \text{in} \quad \mathcal{O}; \\ v = 0 \ \text{ on } S; \quad v = \psi = (\psi^1; \psi^2; \psi^3) \ \text{ on } \Omega, \end{aligned} \tag{13}$$

where $g \in [L^2(\mathcal{O})]^3$ and $\psi \in [L^2(\Omega)]^2 \times \widehat{L}_2(\Omega)$ are given. This type of boundary value problems for the Stokes equation was studied by many authors (see, e.g., [25] and [39] and references therein). We collect some properties of solutions to (13) in the following assertion.

Proposition 2.2 *The following statements hold.*

(1) *Let $g \in [H^{-1+\sigma}(\mathcal{O})]^3$, and $\psi \in [H_*^{1/2+\sigma}(\Omega)]^3$ with $\int_\Omega \psi^3(x')dx' = 0$. Then for every $0 \le \sigma \le 1$ problem (13) has a unique solution $\{v; p\}$ in $[H^{1+\sigma}(\mathcal{O})]^3 \times [H^\sigma(\mathcal{O})/\mathbb{R}]$ such that*

$$\|v\|_{[H^{1+\sigma}(\mathcal{O})]^3} + \|p\|_{H^\sigma(\mathcal{O})/\mathbb{R}} \le c_0 \left\{ \|g\|_{[H^{-1+\sigma}(\mathcal{O})]^3} + \|\psi\|_{[H_*^{\sigma+1/2}(\Omega)]^3} \right\}. \tag{14}$$

(2) *If $g = 0$, $\psi \in [H_*^{-1/2+\sigma}(\Omega)]^3$, $0 \le \sigma \le 1$, $\int_\Omega \psi^3(x')dx' = 0$, then*

$$\|v\|_{[H^\sigma(\mathcal{O})]^3} + \|p\|_{H^{-1+\sigma}(\mathcal{O})/\mathbb{R}} \le c_0 \|\psi\|_{[H_*^{-1/2+\sigma}(\Omega)]^3}. \tag{15}$$

In particular, we can define a linear operator $N_0 : [L^2(\Omega)]^2 \times \widehat{L}_2(\Omega) \mapsto [H^{1/2}(\mathcal{O})]^3$ by the formula

$$N_0\psi = w \quad \textit{iff} \quad \left\{ \begin{array}{l} -\nu\Delta w + \nabla p = 0, \quad \operatorname{div} w = 0 \quad \text{in} \quad \mathcal{O}; \\ w = 0 \quad \text{on } S; \quad w = \psi \quad \text{on } \Omega, \end{array} \right. \tag{16}$$

for $\psi \in [L^2(\Omega)]^2 \times \widehat{L}_2(\Omega)$ ($N_0\psi$ solves (13) with $g \equiv 0$). It follows from (14) and (15) that

$$N_0 : [H_*^s(\Omega)]^2 \times \widehat{H}_*^s(\Omega) \mapsto [H^{1/2+s}(\mathcal{O})]^3 \cap X \quad \textit{continuously}$$

for every $-1/2 \le s \le 3/2$.

Proof. We just combine proof of Proposition 2.1 [9] and Proposition 2.2 [13]. The argument in these references rely on the consideration in [25, 39] and also in [18]. $\square$

3 Well-Posedness Theorem

To define weak (variational) solutions to (1)-(8) we need the following class $\mathcal{L}_T$ of test functions ϕ on $\mathcal{O}$:

$$\mathcal{L}_T = \left\{ \phi \left| \begin{array}{l} \phi \in L_2(0,T; \left[H^1(\mathcal{O})\right]^3), \ \phi_t \in L_2(0,T; [L_2(\mathcal{O})]^3), \\ \operatorname{div}\phi = 0, \ \phi|_S = 0, \ \phi|_\Omega = b = (b^1; b^2; d), \\ d \in L_2(0,T; \widehat{H}_0^2(\Omega)), \ b^j \in L_2(0,T; H_0^1(\Omega)), \ j = 1,2, \\ d_t \in L_2(0,T; \widehat{H}_0^1(\Omega)), \ b_t^j \in L_2(0,T; L_2(\Omega)), \ j = 1,2. \end{array} \right. \right\}.$$

We also denote $\mathcal{L}_T^0 = \{\phi \in \mathcal{L}_T : \phi(T) = 0\}$.

Definition 3.1 A pair of vector functions $(v(t); u(t))$ with $v = (v^1; v^2; v^3)$ and $u = (u^1; u^2; w)$ is said to be a weak solution to the problem in (1)–(9) on a time interval $[0, T]$ if

- $v \in L_\infty(0,T;X) \bigcap L_2(0,T;V)$;
- $u \in L_\infty(0,T; H_0^1(\Omega) \times H_0^1(\Omega) \times H_0^2(\Omega))$;
- $u_t \in L_2(0,T; \left[H_*^{1/2}(\Omega)\right]^3)$ and the compatibility condition $v(t)|_\Omega = (u_t^1; u_t^2; w_t)(t)$ holds for almost all $t \in [0,T]$;
- $(\varrho u_t^1; \varrho u_t^2; w_t) \in L_\infty(0,T; L_2(\Omega) \times L_2(\Omega) \times H_\alpha)$ and $u(0) = u_0$;
- for every $\phi \in \mathcal{L}_T^0$ with $\phi|_\Omega = b = (b^1; b^2; d)$ the following equality holds:

$$
\begin{aligned}
&-\int_0^T (v, \phi_t)_\mathcal{O} dt + \nu \int_0^T E(v, \phi)_\mathcal{O} dt - \int_0^T \left[(M_\alpha w_t, d_t)_\Omega + \varrho(\bar{u}_t, \bar{b}_t)_\Omega\right] dt \\
&+ \gamma \int_0^T (M_\alpha w_t, d)_\Omega dt + \int_0^T (\Delta w, \Delta d)_\Omega dt + \int_0^T a(\bar{u}, \bar{b})_\Omega dt + \int_0^T q(u,b)_\Omega dt \\
&\qquad = (v_0, \phi(0))_\mathcal{O} + (M_\alpha w_1, d(0))_\Omega + \varrho(\bar{u}_1, \bar{b}(0))_\Omega \\
&\qquad\qquad + \int_0^T (G_f(t), \phi)_\mathcal{O} dt + \int_0^T (G_{sh}(t), b)_\Omega dt, \quad (17)
\end{aligned}
$$

where $\bar{u} = (u^1; u^2)$, $\bar{b} = (b^1; b^2)$ and also

$$
\begin{aligned}
E(u, \phi) =& \frac{1}{2} \sum_{i,j=1}^{3} \left(v_{x_i}^j + v_{x_j}^i\right) \left(\phi_{x_i}^j + \phi_{x_j}^i\right), \\
a(\bar{u}, \bar{b}) =& \sum_{i=1}^{2} (\nabla u^i, \nabla b^i)_\Omega + \frac{1+\mu}{1-\mu} (\mathrm{div} u, \mathrm{div} b)_\Omega, \qquad (18) \\
q(u, b) =& (k_1 N_{11} + k_2 N_{22}, d)_\Omega \\
&+ (N_{11} w_{x_1} + N_{12} w_{x_2}, d_{x_1})_\Omega + (N_{12} w_{x_1} + N_{22} w_{x_2}, d_{x_2})_\Omega \\
&+ \frac{1}{1-\mu} \left[((w_{x_1})^2 + \mu (w_{x_2})^2, b_{x_1}^1)_\Omega + ((w_{x_2})^2 + \mu (w_{x_1})^2, b_{x_2}^2)_\Omega\right] \\
&+ (w_{x_1} w_{x_2}, b_{x_2}^1 + b_{x_1}^2)_\Omega + \frac{2}{1-\mu} (w, \kappa_1 b_{x_1}^1 + \kappa_1 b_{x_2}^2)_\Omega.
\end{aligned}
$$

Remark 3.2 **(1)** In the case when we neglect the inertia of longitudinal deformations ($\varrho = 0$) the equations in (6) (or in (10)) become elliptic. However we keep the initial data for the in-plane displacement $(u^1; u^2)$. The point is that the first order evolution for $(u^1; u^2)$ goes from the boundary condition for the fluid velocity in (3).

(2) It also follows from (3) and from the standard trace theorem that for every weak solution $(v(t); u(t))$ we have that

$$
||w(t)||^2_{H_*^{1/2}(\Omega)} + ||u_t^1(t)||^2_{H_*^{1/2}(\Omega)} + ||u_t^2(t)||^2_{H_*^{1/2}(\Omega)} \leq C ||\nabla v(t)||^2_\mathcal{O} \qquad (19)
$$

for almost all $t \in [0,T]$. This estimate does not depend on $\varrho \geq 0$ and provide us an additional regularity estimate for in-plane shell velocities even in the case $\varrho = 0$. Below we use this observation to suggest unified way to prove well-posedness result not distinguishing cases $\varrho > 0$ and $\varrho = 0$ (in contrast with [42], see also [35] ($\varrho = 0$) and [36] ($\varrho > 0$)).

(3) Taking in (17) $\phi(t) = \int_t^T \chi(\tau)d\tau \cdot \psi$, where χ is a smooth scalar function and ψ belongs to the space

$$\widetilde{V} = \left\{ \psi \in V \,\middle|\, \psi|_\Omega = \beta \equiv (\beta^1; \beta^2; \delta) \in H_0^1(\Omega) \times H_0^1(\Omega) \times \widehat{H}_0^2(\Omega) \right\}, \quad (20)$$

one see that the weak solution $(v(t); u(t))$ satisfies the relation

$$\begin{aligned}(v(t), \psi)_{\mathcal{O}} + (M_\alpha w_t(t), \delta)_\Omega + \varrho(\bar{u}_t(t), \bar{\beta})_\Omega \\ = (v_0, \psi)_{\mathcal{O}} + (M_\alpha w_1, \delta)_\Omega + \varrho(\bar{u}_1, \bar{\beta})_\Omega \\ - \int_0^t \big[\nu E(v, \psi) + (\Delta w, \Delta \delta)_\Omega + a(\bar{u}, \bar{\beta}) + \gamma(M_\alpha w_t, \delta)_\Omega \\ + q(u, \beta) - (G_f, \psi)_{\mathcal{O}} - (G_{sh}, \beta)_\Omega\big] d\tau \quad (21)\end{aligned}$$

for all $t \in [0,T]$ and $\psi \in \widetilde{V}$ with $\psi\big|_\Omega = \beta = (\beta^1; \beta^2; \delta)$ and $\bar{\beta} = (\beta^1; \beta^2)$.

As a phase space we use

$$\mathcal{H} = \begin{cases} \{(v_0; u_0; u_1) \in X \times W \times Y : \; v_0 = u_1 \text{ on } \Omega\}, & \varrho > 0, \\ \{(v_0; u_0; w_1) \in X \times W \times H_\alpha : \; (v_0)^3 = w_1 \text{ on } \Omega\} & \varrho = 0 \end{cases} \quad (22)$$

with the norm

$$\|(v_0; u_0; u_1)\|_{\mathcal{H}}^2 = \|v_0\|_{\mathcal{O}}^2 + \|u_0\|_W^2 + \|w_1\|_{H_\alpha}^2 + \varrho \|(u_1^1, u_1^2)\|_\Omega^2,$$

where W and Y are given by (12). We also denote by $\widehat{\mathcal{H}}$ a subspace in $\mathcal{H}$ of the form

$$\widehat{\mathcal{H}} = \left\{ (v_0; u_0; u_1) \in \mathcal{H} : \; w_0 \in \widehat{H}_0^2(\Omega) \right\},$$

where w_0 is the third component of the initial displacement vector u_0.

Our main result in this section is the following well-posedness theorem.

Theorem 3.3 *Assume that $U_0 = (v_0; u_0; u_1) \in \mathcal{H}$, $G_f(t) \in L_2(0,T;V')$, $G_{sh}(t) \in L_2\big(0,T; \big[H^{-1/2}(\Omega)\big]^2 \times H^{-1}(\Omega)\big)$, $\gamma \geq 0$. We also assume that $\varrho \geq 0$ (in the case $\varrho = 0$ the data $\bar{u}_1$ are not fixed). Then for any interval $[0,T]$ there exists a unique weak solution $(v(t); u(t))$ to (1)–(9) with the initial data U_0. This solution possesses the following properties:*

- *In the case $\varrho > 0$ we have*

$$U(t; U_0) \equiv U(t) \equiv (v(t); u(t); u_t(t)) \in C(0,T; X \times W \times Y), \quad (23)$$

If $\varrho = 0$, then

$$U(t; U_0) \equiv U(t) \equiv (v(t); u(t); w_t(t)) \in C(0,T; X \times W \times H_\alpha). \quad (24)$$

- *The solutions depends continuously (both in strong and weak sense) on initial data. More precisely, if $\varrho > 0$ and $U_n \to U_0$ in the norm of $\mathcal{H}$ (resp. weakly in $\mathcal{H}$), then $U(t; U_n) \to U(t; U_0)$ strongly (resp. weakly) in $\mathcal{H}$ for each $t > 0$. In the case $\varrho = 0$ the corresponding convergence take place in $X \times W \times H_\alpha$.*

- *The energy balance equality*

$$\mathcal{E}(v(t), u(t), u_t(t)) + \nu \int_0^t E(v, v) d\tau + \gamma \int_0^t ||w_t||^2_{H_\alpha} d\tau \\ = \mathcal{E}(v_0, u_0, u_1) + \int_0^t (G_f, v)_{\mathcal{O}} d\tau + \int_0^t (G_{sh}, u_t)_\Omega d\tau \quad (25)$$

is valid for every $t > 0$, where the energy functional $\mathcal{E}$ is defined by the relation

$$\mathcal{E}(v, u, u_t) = \frac{1}{2} \left[\|v\|^2_{\mathcal{O}} + \|M_\alpha^{1/2} w_t\|^2_\Omega + \varrho \|\bar{u}_t\|^2_\Omega + \|\Delta w\|^2_\Omega + Q(u) \right] \quad (26)$$

with

$$Q(u) = \frac{2}{(1-\mu)} \int_\Omega \left(\varepsilon_{11}^2 + \varepsilon_{22}^2 + 2\mu \varepsilon_{11}\varepsilon_{22} + \frac{1}{2}(1-\mu)\varepsilon_{12}^2 \right) \\ = \frac{1}{2(1+\mu)} \int_\Omega \left(N_{11}^2 + N_{22}^2 - 2\mu N_{11} N_{22} + 2(1+\mu) N_{12}^2 \right). \quad (27)$$

Proof of Theorem 3.3

We start with the following elliptic type property of the functional $Q(u)$.

Proposition 3.4 *There exists a positive constant C such that for every $u = (u^1; u^2; w) \in W = H_0^1(\Omega) \times H_0^1(\Omega) \times H_0^2(\Omega)$ we have that*

$$||u^1||^2_{1,\Omega} + ||u^2||^2_{1,\Omega} \le C \left[Q(u) + \|w\|^2_\Omega + \|w\|^4_{3/2,\Omega} \right]. \quad (28)$$

Proof. One can see that

$$\left[u^i_{x_i} \right]^2 \le 2\varepsilon_{ii}^2 + 2 \left[k_i w + \frac{1}{2} (w_{x_i})^2 \right]^2, \quad \left[u^1_{x_2} + u^2_{x_1} \right]^2 \le 2\varepsilon_{12}^2 + 2 \left[w_{x_1} w_{x_2} \right]^2.$$

Therefore using the embedding $H^{1/2}(\Omega) \subset L_4(\Omega)$ we obtain that

$$\int_\Omega \left[\left(u^1_{x_1} \right)^2 + \left(u^1_{x_1} \right)^2 + \left(u^1_{x_2} + u^2_{x_1} \right)^2 \right] dx' \le C \left[Q(u) + \|w\|^2_\Omega + \|w\|^4_{3/2,\Omega} \right].$$

Thus, property (28) follows from the Korn inequality (see, e.g., Theorem 3.1 in Chapter 3 of [17]). $\square$

Now we use the compactness method and split the argument into several steps.

Step 1. Existence of an approximate solution for the case $\varrho > 0$. For the construction of Galerkin's approximations in this case we use the same idea as in [13] (which was inspired by the method developed in [8] for the case of a linear plate interacting with nonlinear Navier-Stokes equations).

Let $\{\psi_i\}_{i\in\mathbb{N}}$ be the orthonormal basis in $\widetilde{X} = \{v \in X : v\big|_{\Omega} = 0\}$ consisting of the eigenvectors of the Stokes problem:

$$-\Delta\psi_i + \nabla p_i = \mu_i\psi_i \quad \text{in } \mathcal{O}, \quad \operatorname{div}\psi_i = 0, \quad \psi_i|_{\partial\mathcal{O}} = 0.$$

Here $0 < \mu_1 \leq \mu_2 \leq \cdots$ are the corresponding eigenvalues. Denote by $\{\xi_i\}_{i\in\mathbb{N}}$ the basis in $\widehat{H}^2_0(\Omega)$ which consists of the eigenfunctions of the following problem

$$(\Delta\xi_i, \Delta w)_\Omega = \hat{\kappa}_i(M_\alpha\xi_i, w)_\Omega, \quad \forall\, w \in \widehat{H}^2_0(\Omega),$$

with the eigenvalues $0 < \hat{\kappa}_1 \leq \hat{\kappa}_2 \leq \ldots$ and such that $(M_\alpha\xi_i, \xi_j)_\Omega = \delta_{ij}$. Further, let $\{\eta_i\}_{i\in\mathbb{N}}$ be the basis in $H^1_0(\Omega) \times H^1_0(\Omega)$ which consists of eigenfunctions of the problem

$$a(\eta_i, w) = \tilde{\kappa}_i(\eta_i, w)_\Omega, \qquad \forall\, w \in H^1_0(\Omega) \times H^1_0(\Omega),$$

with the eigenvalues $0 < \tilde{\kappa}_1 \leq \tilde{\kappa}_2 \leq \ldots$ and $||\eta_i||_\Omega = 1$. The form $a(\eta, w)$ is given by (18).

Let $\hat{\phi}_i = N_0(0; 0; \xi_i)$ and $\tilde{\phi}_i = N_0(\eta_i; 0)$, where the operator N_0 is defined by (16). One can see that $\eta_i \in [H^{3/2-\delta}(\Omega)]^2$. Therefore by Proposition 2.2 we have that $\hat{\phi}_i, \tilde{\phi}_i \in [H^{2-\delta}(\mathcal{O})]^3 \cap V$ for every $\delta > 0$.

In what follows we suppose $\phi_i = \tilde{\phi}_i$, $\zeta_i = (\eta_i; 0)$, $\kappa_i = \tilde{\kappa}_i$, and $\gamma_i = 0$; $\phi_{n+i} = \hat{\phi}_i$, $\zeta_{n+i} = (0, 0, \xi_i)$, $\kappa_{n+i} = \hat{\kappa}_i$, and $\gamma_{n+i} = \gamma$ for $i = 1, \ldots, n$. We also define the parameter ϱ_i by the relations: $\varrho_i = \varrho$ for $i = 1, \ldots, n$ and $\varrho_i = 1$ for $i = n+1, \ldots, 2n$.

We define an approximate solution as a pair of functions $(v_{n,m}; u_n)$:

$$\begin{aligned} v_{n,m}(t) &= \sum_{i=1}^{m} \alpha_i(t)\psi_i + \sum_{j=1}^{2n} \dot{\beta}_j(t)\phi_j, \\ u_n(t) &= \sum_{j=1}^{2n} \beta_j(t)\zeta_j + (0; 0; (I - \widehat{P})w_0), \end{aligned} \tag{29}$$

which satisfy the relations

$$\dot{\alpha}_k(t) + \sum_{j=1}^{2n} \ddot{\beta}_j(t)(\phi_j, \psi_k)_{\mathcal{O}} + \nu\mu_k\alpha_k(t) + \nu\sum_{j=1}^{2n} \dot{\beta}_j(t)E(\phi_j, \psi_k)_{\mathcal{O}} = (G_f, \psi_k)_{\mathcal{O}} \tag{30}$$

for $k = 1...m$, and

$$\sum_{i=1}^{m} \dot{\alpha}_i(t)(\psi_i, \phi_k)_{\mathcal{O}} + \sum_{j=1}^{2n} \ddot{\beta}_j(t)(\phi_j, \phi_k)_{\mathcal{O}} + \varrho_k \ddot{\beta}_k(t)$$
$$+ \nu \sum_{i=1}^{m} \alpha_i(t) E(\psi_i, \phi_k)_{\mathcal{O}} + \nu \sum_{j=1}^{2n} \dot{\beta}_j(t) E(\phi_j, \phi_k)_{\mathcal{O}} + \kappa_k \beta_k(t) +$$
$$+ \gamma_k \dot{\beta}_k(t) + q(u_n(t), \zeta_k) = (G_f(t), \phi_k)_{\mathcal{O}} + (G_{sh}(t), \zeta_k)_{\Omega} \quad (31)$$

for $k = 1, \ldots, 2n$. This system of ordinary differential equations is endowed with the initial data

$$v_{v,m}(0) = \Pi_m(v_0 - N_0(0; 0; w_1)) + N_0(0; 0; P_n w_1),$$

$$u_n(0) = (R_n(u_0^1; u_0^2); P_n \widehat{P} w_0 + (I - \widehat{P}) w_0), \ \dot{u}_n(0) = (R_n(u_1^1; u_1^2); P_n w_1),$$

where Π_m is the orthoprojector on $Lin\{\psi_j : j = 1, \ldots, m,\}$ in $\widetilde{X}$, P_n is orthoprojector on $Lin\{\xi_i : i = 1, \ldots, n\}$ in $\widehat{L}_2(\Omega)$ and R_n is orthoprojector on $Lin\{\eta_i : i = 1, \ldots, n\}$ in $L_2(\Omega) \times L_2(\Omega)$. Since Π_m, R_n and P_n are spectral projectors we have that

$$(v_{v,m}(0); u_n(0); \dot{u}_n(0)) \to (v_0; u_0; u_1) \ \text{ strongly in } \mathcal{H} \text{ as } m, n \to \infty. \quad (32)$$

We can rewrite system (30) and (31) as

$$M \frac{d}{dt} \begin{pmatrix} \alpha(t) \\ \dot{\beta}(t) \end{pmatrix} + g(\alpha(t), \beta(t), \dot{\beta}(t)) = G(t)$$

for some locally Lipschitz function $g : \mathbb{R}^{m+4n} \mapsto \mathbb{R}^{m+2n}$, where the function G lies in $L_2(0, T; \mathbb{R}^{m+2n})$ and

$$M = \begin{bmatrix} 0 & 0 \\ 0 & \mathscr{R} \end{bmatrix} + \begin{bmatrix} \{(\psi_i, \psi_j)_{\mathcal{O}}\}_{j,k=1}^{m} & \{(\psi_l, \phi_k)_{\mathcal{O}}\}_{l,k=1}^{m,2n} \\ \{(\phi_k, \psi_l)_{\mathcal{O}}\}_{l,k=1}^{2n,m} & \{(\phi_i, \phi_j))_{\mathcal{O}}\}_{j,k=1}^{2n} \end{bmatrix}, \quad (33)$$

where $\mathscr{R} = \mathrm{diag}\{\varrho_1, \ldots, \varrho_{2n}\}$. The first matrix in (33) is nonnegative and the second one is symmetric and strictly positive (since the functions $\{\psi_i, \phi_j : i = 1, \ldots, m, j = 1, \ldots, 2n\}$ are linearly independent). Therefore system (30) and (31) has a unique solution on some time interval $[0, T']$.

It follows from (29) that

$$v_{n,m}(t) = \sum_{i=1}^{m} \alpha_i(t) \psi_i + N_0[\partial_t u_n(t)],$$

where N_0 is given by (16). This implies the following boundary compatibility condition

$$v_{n,m}(t) = \partial_t u_n(t) \ \text{ on } \ \Omega. \quad (34)$$

Step 2. A priori estimate and limit transition, $\varrho > 0$. Multiplying (30) by $\alpha_k(t)$ and (31) by $\dot{\beta}_k(t)$, after summation we obtain an energy relation of the form (25) for the approximate solutions $(v_{n,m}; u_n)$ (for a similar argument we refer to [13] and also to the classical source [42] on (non-interacting) shell evolution). By Proposition 3.4 and relation (19) this implies the following a priori estimate:

$$\sup_{t\in[0,T]} \|v_{n,m}(t)\|^2_{\mathcal{O}} + \sup_{t\in[0,T]} \left[\|M_\alpha^{1/2}\partial_t w_n(t)\|^2_\Omega + \varrho\|\partial_t \bar{u}_n(t)\|^2_\Omega + \|\Delta w_n(t)\|^2_\Omega + \|\bar{u}_n(t)\|^2_{1,\Omega} \right] + \int_0^T \|\nabla v_{n,m}(t)\|^2_\Omega dt + \int_0^T \|\partial_t \bar{u}_n(t)\|^2_{[H_*^{1/2}(\Omega)]^2} dt \le C_T \quad (35)$$

for any existence interval $[0,T]$ of approximate solutions, where the constant C_T does not depend on n and m. In particular, this implies that any approximate solution can be extended on any time interval by the standard procedure, i.e., the solution is global. It also follows from (35) that the sequence $\{(v_{n,m}; u_n; \partial_t u_n)\}$ contains a subsequence such that

$$(v_{n,m}; u_n; \partial_t u_n) \rightharpoonup (v; u; \partial_t u) \quad *\text{-weakly in } L_\infty(0,T;\mathcal{H}), \quad (36)$$
$$v_{n,m} \rightharpoonup v \quad \text{weakly in } L_2(0,T;V). \quad (37)$$

Moreover, by the Aubin-Dubinsky theorem (see, e.g., [37, Corollary 4]) we can assert that

$$u_n \to u \quad \text{strongly in } C(0,T; H_0^{1-\varepsilon}(\Omega) \times H_0^{1-\varepsilon}(\Omega) \times H_0^{2-\epsilon}(\Omega)) \quad (38)$$

for every $\varepsilon > 0$. Besides, the standard trace theorem yields

$$v_{n,m} \rightharpoonup v \quad \text{weakly in } L_2\left(0,T; \left[H^{1/2}(\partial\mathcal{O})\right]^3\right), \quad (39)$$

thus, we have

$$\partial_t u_n \rightharpoonup \partial_t u \quad \text{weakly in } L_2(0,T; H_*^{1/2}(\Omega)). \quad (40)$$

One can see that $(v_{n,m}; u_n; \partial_t u_n)(t)$ satisfies (17) with the test function ϕ of the form

$$\phi = \phi_{p,q} = \sum_{i=1}^p \gamma_i(t)\psi_i + \sum_{j=1}^q \delta_j(t)\hat{\phi}_j + \sum_{j=1}^q \eta_j(t)\tilde{\phi}_j, \quad (41)$$

where $p \le m$, $q \le n$ and γ_i, δ_j, η_j are scalar absolutely continuous functions on $[0,T]$ with time derivatives from $L_2(0,T)$, such that $\gamma_i(T) = \delta_j(T) = \eta_j(T) = 0$. Using (36) and (37), we can easily pass to the limit in linear terms. Limit transition in the nonlinear terms can be done the same way as

in [42] (see also [33] or [23] for the same type argument). Thus, one can show that $(v; u; \partial_t u)(t)$ satisfies (17) with $\phi = \phi_{p,q}$, where p and q are arbitrary. By (32) and (38) we have $u(0) = u_0$. The compatibility condition (3) follows from (34), (40) and (39).

We conclude the proof of the existence of weak solutions by showing that any function ϕ in $\mathcal{L}_T$ can be approximated by a sequence of functions of the form (41), we refer to [13] for a similar argument in the case of zero in-plane deformations.

This solution satisfies the energy *inequality*:

$$\mathcal{E}(v(t), u(t), u_t(t)) + \nu \int_0^t ||\nabla v||^2_{\mathcal{O}} d\tau + \gamma \int_0^t ||w_t(\tau)||^2_{H_\alpha} d\tau \\ \leq \mathcal{E}(v_0, u_0, u_1) + \int_0^t (G_f(\tau), v)_{\mathcal{O}} d\tau + \int_0^t (G_{sh}(\tau), u_t)_\Omega d\tau$$

for almost all $t > 0$, where the energy functional $\mathcal{E}$ is defined by (26).

Step 3. Existence of weak solutions in the case $\varrho = 0$. We fix some $\bar{u}_1$ from $L_2(\Omega)$ and consider the corresponding solution $(v_\varrho(t); u_\varrho(t))$ with $\varrho > 0$. As above, Proposition 3.4 and relation (19) imply the following a priori estimate:

$$\operatorname*{ess\,sup}_{t\in[0,T]} \left[\|v_\varrho(t)\|^2_{\mathcal{O}} + \|M_\alpha^{1/2} \partial_t w_\varrho(t)\|^2_\Omega + \|\Delta w_\varrho(t)\|^2_\Omega + \|\bar{u}_\varrho(t)\|^2_{1,\Omega} \right] \\ + \int_0^T \|\nabla v_\varrho(t)\|^2_\Omega dt + \int_0^T \|\partial_t \bar{u}_\varrho(t)\|^2_{[H^{1/2}_*(\Omega)]^2} dt \leq C_T \quad (42)$$

for any interval $[0, T]$, where the constant C_T does not depend $\varrho \in (0, 1)$. This implies that the family $\{(v_\varrho; u_\varrho; \partial_t u_\varrho)\}$ contains a subsequence such that

$$\begin{aligned} &(v_\varrho; u_\varrho) \rightharpoonup (v; u) \quad *\text{-weakly in } L_\infty(0, T; X \times W), \\ &\partial_t w_\varrho \rightharpoonup \partial_t w \quad *\text{-weakly in } L_\infty(0, T; H^1_0(\Omega)), \\ &v_\varrho \rightharpoonup v \quad \text{weakly in } L_2(0, T; V), \end{aligned}$$

when $\varrho \to 0$, and also

$$\begin{aligned} &v_\varrho \rightharpoonup v \quad \text{weakly in } L_2(0, T; H^{1/2}(\partial\mathcal{O})), \\ &\partial_t u_\varrho \rightharpoonup \partial_t u \quad \text{weakly in } L_2\left(0, T; \left[H^{1/2}(\partial\mathcal{O})\right]^3\right) \end{aligned}$$

Therefore we have that

$$u_\varrho \to u \quad \text{strongly in } C(0, T; H^{1-\varepsilon}_0(\Omega) \times H^{1-\varepsilon}_0(\Omega) \times H^{2-\epsilon}_0(\Omega))$$

for every $\varepsilon > 0$. This allows us to make limit transition when $\varrho \to 0$ in (17) and prove the existence of weak solutions for the case $\varrho = 0$.

Step 4. Uniqueness. We use the same idea as in [36]. However the additional regularity of $\bar{u}_t$ (see Remark 3.2(2)) makes it possible to cover both cases $\varrho > 0$ and $\varrho = 0$ simultaneously.

Let $(\widehat{v}(t); \widehat{u}(t))$ and $(\widetilde{v}(t); \widetilde{u}(t))$ be two solutions to the problem in question with the same initial data. These solutions satisfy (42) and their difference $(v(t); u(t)) = (\widehat{v}(t) - \widetilde{v}(t); \widehat{u}(t) - \widetilde{u}(t))$ possesses properties

$$\begin{aligned}
&v = (v^1; v^2; v^3) \in L_\infty(0, T; X) \bigcap L_2(0, T; V);\\
&u = (u^1; u^2; w) \in L_\infty(0, T; H_0^1(\Omega) \times H_0^1(\Omega) \times H_0^2(\Omega));\\
&w_t \in L_\infty(0, T; H_\alpha), \quad (u_t^1; u_t^2) \in L_2(0, T; [H_*^{1/2}(\Omega)]^2);
\end{aligned}$$

and by (21) satisfies the relation

$$\begin{aligned}
(v(s), \psi)_{\mathcal{O}} &+ (M_\alpha w_t(s), \delta)_\Omega + \varrho(\bar{u}_t(s), \bar{\beta})_\Omega \\
&= -\int_0^s \big[\nu E(v, \psi) + (\Delta w, \Delta\delta)_\Omega + a(\bar{u}, \bar{\beta}) \\
&\qquad + \gamma(M_\alpha w_t, \delta)_\Omega + q(\widehat{u}, \beta) - q(\widetilde{u}, \beta)\big] d\tau \quad (43)
\end{aligned}$$

for all $s \in [0, T]$ and $\psi \in \widetilde{V}$ with $\psi\big|_\Omega = \beta = (\beta^1; \beta^2; \delta)$ and $\bar{\beta} = (\beta^1; \beta^2)$, where $\widetilde{V}$ is defined by (20).

One can see that $u(t)$ lies in $H_0^1(\Omega) \times H_0^1(\Omega) \times H_0^2(\Omega)$ for each $t \geq 0$ and

$$\psi = i[v](s) \equiv \int_0^s v(\sigma) d\sigma \in \widetilde{V} \quad \text{with} \quad \psi\big|_\Omega = \beta = u(s) \quad \text{for each } s \in [0, T],$$

where we have introduced the notation

$$i[h](s) = \int_0^s h(\sigma) d\sigma, \quad s \in [0, T],$$

for any integrable function $h(t)$ with values in some Banach space.

Substituting this ψ in (43) for the fixed $s \in [0, T]$ after integration from 0 to t over s we obtain that

$$\xi(t) + \varrho\|\bar{u}(t)\|_\Omega^2 \leq 2\left|\int_0^t ds \int_0^s d\tau \left[q(\widehat{u}(\tau), u(s)) - q(\widetilde{u}(\tau), u(s))\right]\right|. \qquad (44)$$

where

$$\begin{aligned}
\xi(t) =& \|i[v](t)\|_{\mathcal{O}}^2 + \|M_\alpha^{1/2} w(t)\|_\Omega^2 + 2\nu \int_0^t ds E\left(i[v](s), i[v](s)\right) \\
&+ 2\gamma \int_0^t ||M_\alpha^{1/2} w(s)||_\Omega^2 ds + \|\Delta i[w](t)\|_\Omega^2 + a\left(i[\bar{u}](t), i[\bar{u}](t)\right).
\end{aligned}$$

Now we estimate the integral term in (44). We first note that the term $q(\widehat{u}(\tau), u(s))$ has the structure

$$q(\widehat{u}(\tau), u(s)) = \sum_{j=1}^4 q_j(\widehat{u}(\tau), u(s)).$$

Here

- $q_1(\widehat{u}, u(s))$ is a linear combination of terms of the form $(\varkappa_0 \widehat{w}, w(s))_\Omega$ and $(\varkappa_{li} D_l \widehat{u}^i, w(s))_\Omega$, where $D_l = \partial_{x_l}$ and $\varkappa_0$, $\varkappa_{li}$ are smooth function, $l, i = 1, 2$;
- $q_2(\widehat{u}, u(s))$ is a linear combination of terms of the form

  $(D_l \widehat{w} D_i \widehat{w}, w(s))_\Omega, \ (D_l \widehat{w} D_i \widehat{w} D_k \widehat{w}, D_m w(s))_\Omega, \ (\varkappa_{lm} \widehat{w} D_l \widehat{w}, D_m w(s))_\Omega;$
- $q_3(\widehat{u}, u(s))$ is a combination of the terms $(D_k \widehat{w}, D_l \widehat{w}, D_m u^i(s))_\Omega$ with $i = 1, 2$;
- $q_4(\widehat{u}, u(s))$ consists of the terms $(D_k \widehat{u}^i, D_l \widehat{w}, D_m w(s))_\Omega$ with $i = 1, 2$.

We also denote

$$\begin{aligned} I_m(t) &= \int_0^t ds \int_0^s d\tau \, [q_m(\widehat{u}(\tau), u(s)) - q_m(\widetilde{u}(\tau), u(s))] \\ &= \int_0^t d\tau \, [q_m(\widehat{u}(\tau), i[u](t) - i[u](\tau)) - q_m(\widetilde{u}(\tau), i[u](t) - i[u](\tau)] \, . \end{aligned}$$

Below the notation $a \sim b_i$ means that a is a linear combination of terms b_i and $a \lesssim b_i$ means that a can be estimated by a linear combination of b_i. With these notations we have

$$\begin{aligned} I_1(t) &\sim \int_0^t ds \int_0^s d\tau \left[(\varkappa_0 w(\tau), w(s))_\Omega) + (\varkappa_{li} D_l u^j(\tau), w(s))_\Omega \right] \\ &= \int_0^t ds \left[(\varkappa_0 i[w](s), w(s))_\Omega) + (\varkappa_{li} D_l i[u^j](s), w(s))_\Omega \right] . \end{aligned}$$

This implies that

$$|I_1(t)| \le C \int_0^t \xi(s) ds, \quad t \in [0, T]. \tag{45}$$

The most complicated term in q_2 is

$$r(\widehat{w}, w(s)) := (D_l \widehat{w} D_i \widehat{w} D_k \widehat{w}, D_m w(s))_\Omega.$$

We consider

$$\begin{aligned} I[r] &= \int_0^t ds \int_0^s d\tau \, [r(\widehat{w}(\tau), w(s)) - r(\widetilde{w}(\tau), w(s))] \\ &\sim \int_0^t d\tau \, (D_l w^*(\tau) D_i w^*(\tau) D_k w(\tau), D_m i[w](t) - D_m i[w](\tau))_\Omega \, , \end{aligned}$$

where w^* is either $\widehat{w}$ or $\widetilde{w}$. Since $H^1(\Omega) \subset L_p(\Omega)$ for every $1 \le p < \infty$

$$\begin{aligned} |I[r]| \le & C \int_0^t \left[\|\widehat{w}(\tau)\|_{2,\Omega}^2 + \|\widetilde{w}(\tau)\|_{2,\Omega}^2 \right] \\ & \qquad \times \|w(\tau)\|_{1,\Omega} \, [\|i[w](t)\|_{2,\Omega} + \|i[w](\tau)\|_{2,\Omega}] \, d\tau \\ \le & \varepsilon \|i[w](t)\|_{2,\Omega}^2 + C_\varepsilon \int_0^t \left[\|w(\tau)\|_{1,\Omega}^2 + \|i[w](\tau)\|_{2,\Omega}^2 \right] d\tau \end{aligned}$$

fore every $\varepsilon > 0$, where the constant C_ε depends on T and on bounds for solutions $\widehat{w}$ and $\widetilde{w}$.

Using a similar argument in other terms of q_2 we obtain the estimate

$$|I_2(t)| \leq \varepsilon\xi(t) + C_\varepsilon \int_0^t \xi(s)ds, \quad t \in [0,T], \quad \forall \varepsilon > 0. \tag{46}$$

To estimate $I_3(t)$ we note that

$$I_3(t) \sim \int_0^t d\tau \left(D_l w^*(\tau) D_k w(\tau), D_m i[u^j](t) - D_m i[u^j](\tau)\right)_\Omega ,$$

where w^* is either $\widehat{w}$ or $\widetilde{w}$ as above. To estimate $I_3(t)$, we use the following Brésis–Gallouet type inequality

$$\|fg\|_\Omega \leq c_1 \left\{\log(1+\sigma)\right\}^{1/2} \|f\|_\Omega \|g\|_{1,\Omega} + \frac{c_2}{1+\sigma} \|f\|_{1,\Omega} \|g\|_{1,\Omega} \tag{47}$$

for every $f, g \in H^1(\Omega)$ and for all $\sigma \geq 0$ (this inequality can be proved the same way as Lemma A.3.6 [12], see also the Appendix in [14]).

Inequality (47) and a priori bound (42) imply

$$\begin{aligned}
\|D_l w^* D_k w\|_\Omega \leq & c_1 \left\{\log(1+\sigma)\right\}^{1/2} \|w^*\|_{2,\Omega} \|w\|_{1,\Omega} + \frac{c_2}{1+\sigma} \|w^*\|_{2,\Omega} \|w\|_{2,\Omega} \\
& \leq C_1(T) \left\{\log(1+\sigma)\right\}^{1/2} \|w\|_{1,\Omega} + \frac{C_2(T)}{1+\sigma}, \quad \forall \sigma > 0.
\end{aligned}$$

Thus,

$$\begin{aligned}
|I_3(t)| \lesssim & \int_0^t d\tau \|D_l w^*(\tau) D_k w(\tau)\|_\Omega \|D_m i[u^j](t) - D_m i[u^j](\tau)\|_\Omega \\
\leq & \frac{C(T)}{1+\sigma} + \varepsilon \|i[u^j](t)\|_{1,\Omega}^2 \\
& + C(\varepsilon, T) \log(1+\sigma) \int_0^t d\tau \left[\|w(\tau)\|_{1,\Omega}^2 + \|i[u^j](\tau)\|_{1,\Omega}^2\right] \\
\leq & \frac{C(T)}{1+\sigma} + \varepsilon\xi(t) + C(\varepsilon, T) \log(1+\sigma) \int_0^t \xi(\tau) d\tau.
\end{aligned} \tag{48}$$

Now we consider $I_4(t)$. First we write it in the form

$$I_4(t) = I_4^a(t) + I_4^b(t),$$

where

$$\begin{aligned}
I_4^a(t) \sim \int_0^t d\tau \left(D_k u^{*j}(\tau) D_l w(\tau), D_m i[w](t) - D_m i[w](\tau)\right)_\Omega , \\
I_4^b(t) \sim \int_0^t d\tau \left(D_k u^j(\tau) D_l w^*(\tau), D_m i[w](t) - D_m i[w](\tau)\right)_\Omega ,
\end{aligned}$$

(the star $*$ in these formulas have the same meaning as above). Using (47) and (42) we have

$$\begin{aligned}|I_4^a(t)| \lesssim & C_T \int_0^t d\tau \|D_l w(\tau)\left(D_m i[w](t) - D_m i[w](\tau)\right)\|_\Omega \\ \leq & C_T \left[\log(1+\sigma)\right]^{1/2} \int_0^t d\tau \|w(\tau)\|_{1,\Omega} \|i[w](t) - i[w](\tau)\|_{2,\Omega} + \frac{C(T)}{1+\sigma} \\ \leq & \frac{C(T)}{1+\sigma} + \varepsilon\xi(t) + C(\varepsilon, T)\log(1+\sigma) \int_0^t \xi(\tau) d\tau. \qquad (49)\end{aligned}$$

Using integration by parts we rewrite term $I_4^b(t)$ in the form

$$I_4^b(t) \sim \int_0^t d\tau (u^j(\tau), D_k \left[D_l w^*(\tau)\left(D_m i[w](t) - D_m i[w](\tau)\right)\right])_\Omega$$

and thus, since $H^{1/2}(\Omega) \subset L_4(\Omega)$, we have

$$\begin{aligned}|I_4^b(t)| \lesssim & \int_0^t d\tau \|u^j(\tau)\|_{1/2,\Omega} \\ & \times \|D_k \left[D_l w^*(\tau)\left(D_m i[w](t) - D_m i[w](\tau)\right)\right]\|_{L_{4/3}(\Omega)}.\end{aligned}$$

One can see that

$$\|D_k(fg)\|_{L_{4/3}(\Omega)} \leq C\|f\|_{1,\Omega}\|g\|_{1,\Omega}, \quad f, g \in H^1(\Omega).$$

Therefore

$$\begin{aligned}|I_4^b(t)| \lesssim & C_T \int_0^t d\tau \|u^j(\tau)\|_{1/2,\Omega} \|i[w](t) - i[w](\tau)\|_{2,\Omega} \\ \leq & \varepsilon \left[\|i[w](t)\|_{2,\Omega}^2 + \int_0^t d\tau \|u^j(\tau)\|_{1/2,\Omega}^2\right] \\ & + C_{T,\varepsilon}\left[\left(\int_0^t d\tau \|u^j(\tau)\|_{1/2,\Omega}\right)^2 + \int_0^t d\tau \|i[w](\tau)\|_{2,\Omega}^2\right].\end{aligned}$$

The trace theorem implies

$$\int_0^t d\tau \|u^j(\tau)\|_{1/2,\Omega}^2 \leq C \int_0^t ds E\left(i[v](s), i[v](s)\right),$$

and thus

$$|I_4^b(t)| \leq \left[\varepsilon + C_{T,\varepsilon} t\right]\xi(t) + C_{T,\varepsilon}\int_0^t d\tau \xi(\tau), \quad t \in [0, T] \qquad (50)$$

The estimates in (45), (46), (48), (49), (50) and (44) allow us to choose ε and $T_* > 0$ such that

$$\xi(t) \leq \frac{C_1}{1+\sigma} + C_2 \log(1+\sigma) \int_0^t \xi(\tau) d\tau \quad \text{for all} \quad t \in [0, T_*]$$

with arbitrary $\sigma > 1$. Now as in [35, 36] (see also [12, Appendix A]) applying Gronwall's lemma we can conclude that $\xi(t) \equiv 0$ for all $t \in [0, T_{**}]$ for some $T_{**} \leq T_*$.

Remark 3.5 In the case $\alpha = 0$ we can prove the existence of weak solutions which satisfies the energy inequality, using the same type of argument. The uniqueness of these solutions is still an open question. Sedenko's method does not work here because the nonlinearity is strongly supercritical when $\alpha = 0$.

Step 5. Continuity with respect to t and the energy equality. First we note that the vector $(v(t); u(t); \varrho u_t^1(t); \varrho u_t^2(t); w_t(t))$ is weakly continuous in $\mathcal{H}$ for any weak solution $(v(t); u(t))$. Indeed, it follows from (21) that $(v(t); u(t))$ satisfies the relation

$$(v(t), \psi)_{\mathcal{O}} = (v_0, \psi(0))_{\mathcal{O}} + \int_0^t [-\nu E(v, \psi) + (G_f(\tau), \psi)_{\mathcal{O}}]\, d\tau$$

for almost all $t \in [0, T]$ and for all $\psi \in V_0 = \{v \in V : v|_\Omega = 0\} \subset \widetilde{V} \subset V$, where $\widetilde{V}$ is given by (20). This implies that $v(t)$ is weakly continuous in V_0'. Since $X \subset V_0'$, we can apply Lions lemma (see [31, Lemma 8.1]) and conclude that $v(t)$ is weakly continuous in X. The same lemma gives us weak continuity of $u(t)$ in W. Now using (21) again with $\psi \in \widetilde{V}$ we conclude that

$$t \mapsto (M_\alpha w_t(t), \delta)_\Omega + \varrho(\bar{u}_t(t), \bar{\beta})_\Omega \quad \text{is continuous}$$

for every $\beta = (\bar{\beta}; \delta) \in H_0^1(\Omega) \times H_0^1(\Omega) \times \widehat{H}_0^2(\Omega)$. This imply that

$$t \mapsto (\varrho u_t^1(t); \varrho u_t^2(t); w_t(t)) \quad \text{is weakly continuous}$$

in $Y = L_2(\Omega) \times L_2(\Omega) \times \widehat{H}_0^1(\Omega)$ for every $\varrho \geq 0$.

In the proof the energy equality we follow the scheme of [23]. To this end we need to introduce finite difference operator D_h, depending on a small parameter h.

Let g be a bounded function on $[0, T]$ with values in some Hilbert space. We extend $g(t)$ for all $t \in \mathbb{R}$ by defining $g(t) = g(0)$ for $t < 0$ and $g(t) = g(T)$ for $t > T$. With this extension we denote

$$g_h^+(t) = g(t+h) - g(t), \qquad g_h^-(t) = g(t) - g(t-h),$$
$$D_h g(t) = \frac{1}{2h}(g_h^+(t) + g_h^-(t)).$$

Properties of the operator D_h are collected in Proposition 4.3 [23].

Using weak continuity of weak solutions, we can extend the variational relation in (17) on the class of test functions from $\mathcal{L}_T$ (instead of $\mathcal{L}_T^0$) by an appropriate limit transition. More precisely, one can show that any weak

solution $(v;u)$ (with $u=(u^1;u^2;w)\equiv(\bar{u};w)$) satisfies the relation

$$
\begin{aligned}
-\int_0^T(v,\phi_t)_{\mathcal{O}}dt+\nu\int_0^T E(v,\phi)_{\mathcal{O}}dt-\int_0^T\big[(M_\alpha w_t,d_t)_\Omega+\varrho(\bar{u}_t,\bar{b}_t)_\Omega\big]dt \\
+\gamma\int_0^T(M_\alpha w_t,d)_\Omega dt+\int_0^T(\Delta w,\Delta d)_\Omega dt+\int_0^T a(\bar{u},\bar{b})_\Omega dt+\int_0^T q(u,b)_\Omega dt \\
=(v_0,\phi(0))_{\mathcal{O}}+(M_\alpha w_1,d(0))_\Omega+\varrho(\bar{u}_1,\bar{b}(0))_\Omega \\
-\big[(v(T),\phi(T))_{\mathcal{O}}+(M_\alpha w_t(T),d(T))_\Omega+\varrho(\bar{u}_t(T),\bar{b}(T))_\Omega\big] \\
+\int_0^T(G_f(t),\phi)_{\mathcal{O}}dt+\int_0^T(G_{sh}(t),b)_\Omega dt
\end{aligned}
\tag{51}
$$

for every $\phi\in\mathcal{L}_T$ with $\phi\big|_\Omega=b=(b^1;b^2;d)\equiv(\bar{b};d)$.

Now we use

$$
\phi=\frac{1}{2h}\int_{t-h}^{t+h}v(\tau)d\tau
$$

as a test function in (51). For the shell component we have test function $b=\phi|_\Omega=D_hu$ – the same one that used in [23] for the full Karman model. Thus, all the arguments for the shell component in our model are the same as in [23], and we need to treat the fluid component only. Using Proposition 4.3 [23], one can conclude, that

$$
\int_0^T dt\left[(v(t),D_hv(t))_{\mathcal{O}}+\nu E\left(v(t),\frac{1}{2h}\int_{t-h}^{t+h}v(\tau)d\tau\right)\right]\to
\frac{1}{2}\left[v(T)-v(0)\right]+\nu\int_0^T dtE(v(t),v(t))
$$

when $h\to 0$. This makes it possible to prove the energy equality in (25).

Continuity of weak solutions with respect to t stated in (23) and (24) can be obtained in the standard way from the energy equality and weak continuity (see [31, Ch. 3] and also [23]).

Step 6. Continuity with respect to initial data. First we prove the continuity with respect to weak topology.

Let $\varrho>0$ and $\{U_0^n\}$ be the sequence of initial data such that $U_0^n\rightharpoonup U_0$ weakly in $\mathcal{H}$ (defined by (22)) as $n\to\infty$. We need to prove that

$$
U^n(t)\rightharpoonup U(t)\quad\text{weakly in }\ \mathcal{H}\ \text{ for every }\ t>0,
$$

where $U^n(t)=(v^n(t);u^n(t);u_t^n(t))$ and $U(t)=(v(t);u(t);u_t(t))$ are weak solutions to the problem in question with the corresponding initial data. Using energy relation (25) and Proposition 3.4, we conclude that

$$
\sup_{[0,T]}||U^n(t)||^2_{\mathcal{H}}\le C_T,\quad n=1,2,\dots
$$

This enables us to extract a subsequence such that

$$U^n(t) \rightharpoonup U^*(t) \quad *\text{-weakly in } L_\infty(0,T;\mathcal{H}); \tag{52}$$

$$v^n(t) \rightharpoonup v^*(t) \quad \text{weakly in } L_2(0,T;V); \tag{53}$$

$$w_t^n(t) \rightharpoonup w_t^*(t) \quad \text{weakly in } L_2(0,T;H_\alpha); \tag{54}$$

for some $U^*(t) = (v^*(t); u^*(t); u_t^*(t))$. Performing limit transition in (17) and using weak convergence of initial data, we obtain that $(v^*; u^*)$ is a weak solution and thus $U^*(t) = U(t)$ by the uniqueness statement.

To proceed with the proof, we need to establish additional estimates for time derivatives.

Taking $\psi \in \mathcal{L}_T^0$ such that $\psi(0) = 0$ and $\psi|_\Omega = 0$ as a test function in (17), we obtain

$$\int_0^T (v^n, \psi_t)_{\mathcal{O}} d\tau = \int_0^T [E(v^n, \psi) - (G_f, \psi)_{\mathcal{O}}]\, d\tau,$$

which allows to define v_t^n as a functional on $V_0 = \{\phi \in V : \ \phi|_\Omega = 0\}$ and get the estimate

$$||v_t^n||_{L_2(0,T;(V_0)')} \le C(T).$$

Using $N_0 b$ with $b = (b^1; b^2; d) \in C_0^1(0,T; H_0^1(\Omega) \times H_0^1(\Omega) \times \times \widehat{H}_0^2(\Omega))$ as a test function, we obtain the estimate

$$||(1-\alpha\Delta)w_{tt}||_{L_2(0,T;\mathcal{F}_{-2}(\Omega))} + \varrho||\bar{u}_{tt}||_{L_2(0,T;H^{-1}(\Omega))} \le C(T),$$

where $\mathcal{F}_{-s}(\Omega)$ with $s \in [1,2]$ is adjoint to $H_0^s(\Omega)$ with respect to duality generated by $H_\alpha(\Omega)$. Thus we have the following additional convergence properties:

$$v_t^n \rightharpoonup v_t \qquad \text{weakly in } L_2(0,T;[H^{-1}(\mathcal{O})]^3),$$
$$\varrho\bar{u}_{tt}^n \rightharpoonup \varrho\bar{u}_{tt} \qquad \text{weakly in } L_2(0,T;[H^{-1}(\Omega)]^2),$$
$$w_{tt}^n \rightharpoonup w_{tt} \qquad \text{weakly in } L_2(0,T;\mathcal{F}_{-2}(\Omega)),$$

We note that $H_\alpha(\Omega) \subset H_0^\sigma(\Omega) \subset L_2(\Omega) \subset \mathcal{F}_{-2}(\Omega)$ for every $\sigma \in [0,1]$. Therefore from the relations above and also from (52)–(54) using the Aubin-Dubinsky theorem (see [37]) we conclude that

$$v^n \to v \qquad \text{strongly in } C(0,T;H^{-\varepsilon}),$$
$$u^n \to u \qquad \text{strongly in } C(0,T;H_0^{1-\varepsilon}(\Omega) \times H_0^{1-\varepsilon}(\Omega) \times H_0^{2-\varepsilon}(\Omega)),$$
$$w_t^n \to w_t \qquad \text{strongly in } C(0,T;H_0^{1-\varepsilon}(\Omega)),$$
$$\varrho\bar{u}_t^n \to \varrho\bar{u}_t \qquad \text{strongly in } C(0,T;[H^{-\varepsilon}(\Omega)]^2).$$

In particular, this implies that $U^n(t) \to U^*(t)$ weakly in $\mathcal{H}$ when $U_0^n \to U_0$.

To prove strong continuity with respect to initial data, we use an idea borrowed from [23]. Due to weak continuity already established we need

only to show the convergence of $\|U^n(t)\|_{\mathcal{H}}$ to $\|U(t)\|_{\mathcal{H}}$ for each $t > 0$ under the condition $\|U_0^n - U_0\|_{\mathcal{H}} \to 0$ as $n \to \infty$.

Let $Q_0(u)$ be given by the first line of (27) with

$$\varepsilon_{11} = \varepsilon_{11}^0 \equiv u_{x_1}^1, \quad \varepsilon_{22} = \varepsilon_{22}^0 \equiv u_{x_2}^2, \quad \varepsilon_{12} = \varepsilon_{12}^0 \equiv u_{x_2}^1 + u_{x_1}^2$$

Due to the argument given in the proof of Proposition (3.4) $\sqrt{Q_0(u)}$ is an equivalent norm on $H_0^1(\Omega) \times H_0^1(\Omega)$. Thus, to establish convergence of $U^n(t)$ to $U(t)$ in $\mathcal{H}$ it is sufficient to show that

$$\mathcal{E}_0(U^n(t)) \to \mathcal{E}_0(U(t)) \quad \text{as } n \to \infty \quad \text{for every} \quad t > 0,$$

where

$$\mathcal{E}_0(v, u, u_t) = \frac{1}{2}\left[\|v\|_{\mathcal{O}}^2 + \|M_\alpha^{1/2} w_t\|_\Omega^2 + \varrho\|\bar{u}_t\|_\Omega^2 + \|\Delta w\|_\Omega^2 + Q_0(u)\right]. \quad (55)$$

To prove this we use the energy relation in (25). First we note that the potential energy term $Q(u)$ in the energy $\mathcal{E}$ given in (26) has the form

$$Q(u) = Q_0(u) + \mathrm{Comp}\,(u), \quad (56)$$

where $\mathrm{Comp}\,(u)$ is a functional which is continuous with respect to weak convergence in $\mathcal{H}$. Since $\mathcal{E}(U_0^n) \to \mathcal{E}(U_0)$, it follows from energy equality (25) that

$$\lim_{n\to\infty} [\mathcal{E}_0(U^n(t)) + \Phi_t(U^n)] = \mathcal{E}_0(U(t)) + \nu \int_0^t E(v, v) d\tau + \Phi_t(U),$$

where

$$\Phi_t(U) = \nu \int_0^t E(v, v) d\tau + \gamma \int_0^t (M_\alpha w_\tau, w_\tau) d\tau$$

Using lower semicontinuity of the functional $\Phi_t(U)$ with respect to weak convergence $U^n(t) \rightharpoonup U(t)$ in $L_2(0, T; \mathcal{H})$, we obtain that

$$\liminf_{n\to\infty} \mathcal{E}_0(U^n(t)) \le \mathcal{E}_0(U(t)),$$

which together with lower semicontinuity of the (quadratic) energy $\mathcal{E}_0$ gives us the desired result. This completes the proof of Theorem 3.3.

4 Stationary solutions

In this section following ideas presented in [13] and [41] we describe properties of stationary solutions in the case when the forces G_f and G_{sh} are autonomous. These solutions are the same in both cases $\varrho > 0$ and $\varrho = 0$.

It follows from Definition 3.1 that a stationary solution $(v; u) \in V \times W$ satisfies the relation

$$\nu E(v, \psi) + (\Delta w, \Delta \delta)_\Omega + a(\bar{u}, \bar{\beta}) \\ + q(u, \beta) - (G_f, \psi)_{\mathcal{O}} - (G_{sh}, \beta)_\Omega = 0 \quad (57)$$

for all $\psi \in \widetilde{V}$ with $\psi\big|_{\Omega} = \beta = (\beta^1; \beta^2; \delta)$ and $\bar{\beta} = (\beta^1; \beta^2)$. The space $\widetilde{V}$ is given by (20), i.e.,

$$\widetilde{V} = \left\{ \psi \in V \;\middle|\; \psi|_{\Omega} = \beta \equiv (\beta^1; \beta^2; \delta) \in \widehat{W} \right\},$$

where

$$\widehat{W} = H_0^1(\Omega) \times H_0^1(\Omega) \times \widehat{H}_0^2(\Omega). \tag{58}$$

Moreover, by the comparability condition we have that $v|_{\partial\mathcal{O}} = 0$.

We start with the following description of stationary solutions.

Proposition 4.1 *A couple $(v; u) \in V \times W$ is a stationary solution if and only if*

- *v lies in $V_0 = \{\psi \in V : \psi|_{\partial\mathcal{O}} = 0\}$ and satisfies the relation*

$$\nu E(v, \psi) - (G_f, \psi)_{\mathcal{O}} = 0, \quad \forall \psi \in V_0; \tag{59}$$

- *$u = (u^1; u^2; w) \in W = H_0^1(\Omega) \times H_0^1(\Omega) \times H_0^2(\Omega)$ satisfies the equality*

$$(\Delta w, \Delta\delta)_{\Omega} + \bar{q}(u, \beta) - (G_{sh} + N_0^* G_f, \beta)_{\Omega} = 0 \tag{60}$$

for all $\beta = (\beta^1; \beta^2; \delta) \in \widehat{W}$, where $\widehat{W}$ is given by (58) and

$$\begin{aligned} \bar{q}(u, \beta) = &(k_1 N_{11} + k_2 N_{22}, \delta)_{\Omega} \\ &+ (N_{11} w_{x_1} + N_{12} w_{x_2}, \delta_{x_1})_{\Omega} + (N_{12} w_{x_1} + N_{22} w_{x_2}, \delta_{x_2})_{\Omega} \\ &+ (N_{11}, \beta^1_{x_1})_{\Omega} + (N_{12}, \beta^1_{x_2} + \beta^2_{x_1})_{\Omega} + (N_{22}, \beta^2_{x_2})_{\Omega} \end{aligned} \tag{61}$$

Proof. Taking $\beta \equiv 0$ in (57) yields (59). Since $v\big|_{\partial\mathcal{O}} = 0$, one can see that $E(v, N_0\beta) = 0$ for every $\beta \in \widehat{W}$. Therefore taking $\psi = N_0\beta$ in (57) gives us (60). Thus any stationary solution satisfies (59) and (60). Similarly, we can derive (57) from (59) and (60). $\square$

The following assertion shows that for the forces G_f and G_{sh} of a special structure we can guarantee the existence of stationary solutions.

Proposition 4.2 *We assume that*

$$G_f \equiv 0, \quad G^1_{sh} = G^2_{sh} \equiv 0 \;\; \text{and} \;\; G^3_{sh} \equiv g \in H^{-1}(\Omega). \tag{62}$$

Then any stationary solution has the form $(0; u)$, where $u = (u^1; u^2; w) \in W$ satisfies

$$(\Delta w, \Delta\delta)_{\Omega} + \bar{q}(u, \beta) - (g, \delta)_{\Omega} = 0, \quad \forall \beta = (\beta^1; \beta^2; \delta) \in \widehat{W}. \tag{63}$$

Moreover, there exists at least one solution $u = (u^1; u^2; w)$ to (63) in the space $\widehat{W}$. If in addition we assume that $k_1 = k_2 = 0$, then the set of all stationary solutions to (63) from $\widehat{W}$ is bounded in the space $\widehat{W}$.

Proof. If $G_f \equiv 0$, then it follows from (59) that $v \equiv 0$. Thus equation (60) turns into (63).

To prove the existence of elements $u = (u^1; u^2; w) \in \widehat{W}$ satisfying (63), we note (see, e. g. [41]) that a solution can be obtained as a minimum point of the functional

$$\Pi(u) = \frac{1}{2}\left[\|\Delta w\|^2 + Q(u)\right] - (w, g)_\Omega \quad \text{on} \quad \widehat{W},$$

where Q is given by (27). It is clear that $\Pi(u)$ is bounded from below.[2] Thus we can construct appropriate Galerkin approximations $\{u_n\} \subset \widehat{W}$ for the global minimum and note that $\Pi(u_n) \le \Pi(0)$. This facts together with Proposition 3.4 provide us an a priori estimate which allows to prove the existence of a solution by the same method as in [41].

To prove the boundedness of stationary solutions we consider the functional $\bar{q}(u, \beta)$ given by (61) with $\beta = (\beta^1; \beta^2; \delta)$, where with $\delta = \frac{1}{2}w$ and $\beta^i = u^i$. In this case we obtain that

$$\begin{aligned}\bar{q}(u,\beta) = &-\frac{1}{2}(k_1 N_{11} + k_2 N_{22}, w)_\Omega + (N_{11}, u^1_{x_1} + k_1 w + \frac{1}{2}[w_{x_1}]^2)_\Omega \\ &+ (N_{12}, u^1_{x_2} + u^2_{x_1} + w_{x_1} w_{x_2})_\Omega + (N_{22}, u^2_{x_2} + k_2 w + \frac{1}{2}[w_{x_2}]^2)_\Omega.\end{aligned}$$

Using the expressions for the deformation tensor $\{\varepsilon_{ij}\}$ we obtain

$$\begin{aligned}\bar{q}(u,\beta) = &-\frac{1}{2}(k_1 N_{11} + k_2 N_{22}, w)_\Omega \\ &+ (N_{11}, \varepsilon_{11})_\Omega + (N_{12}, \varepsilon_{12})_\Omega + (N_{22}, \varepsilon_{22})_\Omega.\end{aligned}$$

The Hooke law (4) yields

$$\begin{aligned}\bar{q}(u,\beta) &= -\frac{1}{2}(k_1 N_{11} + k_2 N_{22}, w)_\Omega \\ &\quad + \frac{2}{(1-\mu)}\int_\Omega \left(\varepsilon_{11}^2 + \varepsilon_{22}^2 + 2\mu\varepsilon_{11}\varepsilon_{22} + \frac{1}{2}(1-\mu)\varepsilon_{12}^2\right) \\ &= -\frac{1}{2}(k_1 N_{11} + k_2 N_{22}, w)_\Omega + Q(u),\end{aligned}$$

where $Q(u)$ is given by (27). Therefore under the conditions in (62) from (63) we have that

$$\frac{1}{2}\|\Delta w\|^2_\Omega + Q(u) - \frac{1}{2}(k_1 N_{11} + k_2 N_{22}, w)_\Omega = \frac{1}{2}(g, w)_\Omega$$

Thus, the set of stationary solutions is bounded provided $k_i = 0$ (or even small enough). $\square$

To the best of our knowledge the existence of stationary solutions in the case of general external loads is still an open question.

[2] This is exactly the point where we use the structure of the external forces assumed in (62)

5 Existence of a global attractor

In this section we prove the existence of a compact global attractor under the condition that the external forces satisfy (62).

First we note that by Theorem 3.3 problem (1)–(9) generates an evolution semigroup S_t in the space $\widehat{\mathcal{H}}$, which has the form

- $\widehat{\mathcal{H}} = X \times \widehat{W} \times Y$ in the case $\varrho > 0$,
- $\widehat{\mathcal{H}} = X \times \widehat{W} \times H_\alpha$ in the case $\varrho = 0$,

where X and Y are defined in (11) and (12) and $\widehat{W}$ is given by (58). The evolution operator S_t is defined as follows

- **Case** $\varrho > 0$**:** $S_t(v_0; u_0; u_1) \equiv U(t) = (v(t); u(t); u_t(t))$, where the couple $(v(t); u(t))$ solves (1)–(9).
- **Case** $\varrho = 0$**:** $S_t(v_0; u_0; w_1) \equiv \bar{U}(t) = (v(t); u(t); w_t(t))$, where $v(t)$ and $u(t) = (u^1(t); u^2(t); w(t))$ solves (1)–(9) with $\varrho = 0$.

Our main result in this section is the following theorem.

Theorem 5.1 *Assume that $\gamma > 0$ and the external forces satisfy* (62). *Let the set of the stationary points in $\widehat{\mathcal{H}}$ of the problem* (1)–(9) *is bounded. Then the evolution semigroup S_t generated by this problem possesses a compact global attractor.*

We recall (see, e.g., [4, 10, 38]) that *global attractor* of the dynamical system $(S_t, \widehat{\mathcal{H}})$ is defined as a bounded closed set $\mathfrak{A} \subset \widehat{\mathcal{H}}$ which is invariant ($S_t\mathfrak{A} = \mathfrak{A}$ for all $t > 0$) and uniformly attracts all other bounded sets:

$$\lim_{t\to\infty} \sup\{\mathrm{dist}_{\widehat{\mathcal{H}}}(S_t y, \mathfrak{A}) : \ y \in B\} = 0 \quad \text{for any bounded set } B \text{ in } \widehat{\mathcal{H}}.$$

Proof. It follows from energy inequality in (25) that the set

$$\mathcal{W}_R = \{U \ : \ \Phi(U) \equiv \mathcal{E}(U) - (g, w)_\Omega \le R\}$$

is forward invariant with respect to S_t for each $R > 0$. Here $U = (v; u; u_t)$ with $u = (u^1; u^2; w)$ in the case $\varrho > 0$ and $U = (v; u; w_t)$ in the case $\varrho = 0$. As in the proof of Proposition 4.2 using (62) one can see that $\Phi(U^n) \to +\infty$ if and only if $\|U^n\|_{\mathcal{H}} \to +\infty$. Therefore the set $\mathcal{W}_R$ is bounded and any bounded set belongs to $\mathcal{W}_R$ for some R. Moreover, it follows from energy inequality (25) that the continuous functional $\Phi(U)$ on $\widehat{\mathcal{H}}$ possesses the properties (i) $\Phi\big(S_tU\big) \le \Phi(U)$ for all $t \ge 0$ and $U \in \widehat{\mathcal{H}}$; (ii) the equality $\Phi(U) = \Phi(S_tU)$ holds for all $t > 0$ only if U is a stationary point of S_t. This means that $\Phi(U)$ is a *strict Lyapunov function* and $(\widehat{\mathcal{H}}, S_t)$ is a gradient dynamical system. Therefore due to Corollary 2.29[11] (see also Theorem 4.6 in [34]) we need only to prove asymptotic smoothness. We recall that (see,

e.g., [22] or [11]) that a dynamical system (X, S_t) is said to be *asymptotically* smooth if for any closed bounded set $B \subset X$ that is positively invariant ($S_t B \subseteq B$) one can find a compact set $\mathcal{K} = \mathcal{K}(B)$ which uniformly attracts B, i. e. $\sup\{\mathrm{dist}_X(S_t y, \mathcal{K}) :\ y \in B\} \to 0$ as $t \to \infty$.

To prove asymptotic smoothness we use Ball's method of energy relations (see [5] and also [32]). For a convenience we recall the abstract theorem (in a slightly relaxed form) from [32] which represents the main idea of the method.

Theorem 5.2 ([32]) *Let S_t be a semigroup of strongly continuous operators in some Hilbert space $\mathcal{H}$. Assume that operators S_t are also weakly continuous in $\mathcal{H}$ and there exist a number $\omega > 0$ and functionals Λ, L and K on $\mathcal{H}$ such that the equality*

$$\Lambda(t) + \int_s^t L(\tau) e^{-2\omega(t-\tau)} d\tau = \Lambda(s) e^{-2\omega(t-s)} + \int_s^t K(\tau) e^{-2\omega(t-\tau)} d\tau, \quad (64)$$

holds on the trajectories of the system $(\mathcal{H}, S_t)$. Here we use the notation $G(t) = G(S_t U)$, where G is one of the symbols Λ, L and K.

Let the functionals possess the properties:

(i) *$\Lambda : \mathcal{H} \to \mathbb{R}_+$ is a continuous bounded functional and if $\{U_j\}_j$ is bounded sequence in $\mathcal{H}$ and $t_j \to +\infty$ is such that (a) $S_{t_j} U_j \rightharpoonup U$ weakly in $\mathcal{H}$, and (b) $\limsup_{n\to\infty} \Lambda(S_{t_j} U_j) \le \Lambda(U)$, then $S_{t_j} U_j \to U$ strongly in $\mathcal{H}$.*

(ii) *$K :\ \mathcal{H} \to \mathbb{R}$ is 'asymptotically weakly continuous' in the sense that if $\{U_j\}_j$ is bounded in $\mathcal{H}$, and $S_{t_j} U_j \rightharpoonup U$ weakly in $\mathcal{H}$ as $t_j \to +\infty$, then $K(S_\tau U) \in L_1^{loc}(\mathbb{R}_+)$ and*

$$\lim_{j\to\infty} \int_0^t e^{-2\omega(t-s)} K(S_{s+t_j} U_j) ds = \int_0^t e^{-2\omega(t-s)} K(S_s U) ds, \quad \forall t > 0. \quad (65)$$

(iii) *L is 'asymptotically weakly lower semicontinuous' in the sense that if $\{U_j\}_j$ is bounded in $\mathcal{H}$, $t_j \to +\infty$, $S_{t_j} U_j \rightharpoonup U$ weakly in $\mathcal{H}$, then $L(S_\tau U) \in L_1^{loc}(\mathbb{R}_+)$ and*

$$\liminf_{j\to\infty} \int_0^t e^{-2\omega(t-s)} L(S_{s+t_j} U_j) ds \ge \int_0^t e^{-2\omega(t-s)} L(S_s U) ds, \quad \forall t > 0.$$

Then S_t is asymptotically smooth.

Now we check validity of the hypotheses of Theorem 5.2 in our case.

Step 1 (continuity): The continuity of the evolutional operator in both (strong and weak) senses follows from Theorem 3.3.

Step 2 (energy equality): All the calculations below can be justified by considering approximate solutions.

Let $\Psi(t) = \Psi(v(t), u(t), u_t(t))$, where

$$\Psi(v, u, u_t) = (M_\alpha w_t, w)_\Omega + \varrho(\bar{u}_t, \bar{u})_\Omega + (v, N_0[u])_{\mathcal{O}},$$

where $U = (v; u; u_t) \equiv (v; \bar{u}; w; \bar{u}_t; w_t)$ is a weak solution and N_0 is given by (16). One can see that

$$\begin{aligned}\frac{d}{dt}\Psi(t) =& (M_\alpha w_{tt}, w)_\Omega + \varrho(\bar{u}_{tt}, \bar{u})_\Omega + (v_t, N_0[u])_{\mathcal{O}} \\ &+ \|M_\alpha^{1/2} w_t\|_\Omega^2 + \varrho\|\bar{u}_t\|_\Omega^2 + (v, N_0[u_t])_{\mathcal{O}} \\ =& \|M_\alpha^{1/2} w_t\|_\Omega^2 + \varrho\|\bar{u}_t\|_\Omega^2 + (v, N_0[u_t])_{\mathcal{O}} - \nu E(v, N_0[u]) \\ &- \|\Delta w\|^2 - Q(u) - \frac{1}{1-\mu}\int_\Omega \left[w_{x_1}^4 + w_{x_2}^4 + (1+\mu) w_{x_1}^2 w_{x_2}^2\right] dx' \\ &- \gamma(M_\alpha w_t, w)_\Omega + \Phi_0(t) + (g, w)_\Omega,\end{aligned}$$

where $\Phi_0(t)$ is a linear combination of the terms of the form

$$(D_l u^i D_j w, D_m w)_\Omega \quad \text{and} \quad (k_i w D_l w, D_m w)_\Omega.$$

Let $\Lambda(t) = \mathcal{E}(t) + \eta\Psi(t)$, where $\eta > 0$ will be chosen later. Since

$$\frac{d}{dt}\mathcal{E}(t) = -\nu E(v, v) - \gamma||M_\alpha^{1/2} w_t||_\Omega^2 + (g, w_t)_\Omega$$

one can see

$$\begin{aligned}&\frac{d}{dt}\Lambda(t) + (\gamma - \eta)\|M_\alpha^{1/2} w_t\|_\Omega^2 + \nu E(v, v) - \eta\varrho\|\bar{u}_t\|_\Omega^2 \\ &\quad + \eta\left[\|\Delta w\|^2 + Q(u) + \frac{1}{1-\mu}\int_\Omega \left[w_{x_1}^4 + w_{x_2}^4 + (1+\mu) w_{x_1}^2 w_{x_2}^2\right] dx'\right] \\ &= \Phi_1(t),\end{aligned}$$

where

$$\begin{aligned}\Phi_1(t) =& \eta\left[(v, N_0[u_t])_{\mathcal{O}} - \nu E(v, N_0[u]) - \gamma(M_\alpha w_t, w)_\Omega\right] \\ &+ \eta\Phi_0(t) + \eta(g, w)_\Omega + (g, w_t)_\Omega.\end{aligned}$$

Consequently

$$\begin{aligned}&\frac{d}{dt}\Lambda + 2\omega\Lambda + (\gamma - \eta - \omega)\|M_\alpha^{1/2} w_t\|_\Omega^2 + (\nu - \omega)E(v, v) - (\eta + \omega)\varrho\|\bar{u}_t\|_\Omega^2 \\ &\quad + (\eta - \omega)\left[\|\Delta w\|^2 + Q(u)\right] \\ &\quad + \frac{\eta}{1-\mu}\int_\Omega \left[w_{x_1}^4 + w_{x_2}^4 + (1+\mu) w_{x_1}^2 w_{x_2}^2\right] dx' = \Phi_1(t) + \omega\eta\Psi(t).\end{aligned}$$

Thus we obtain (64) with L and K given by

$$\begin{aligned}L(t) =& (\gamma - \eta - \omega)\|M_\alpha^{1/2} w_t\|_\Omega^2 + (\nu - \omega)E(v, v) \\ &- (\eta + \omega)\varrho\|\bar{u}_t\|_\Omega^2 + (\eta - \omega)\left[\|\Delta w\|^2 + Q(u)\right] \\ &+ \frac{\eta}{1-\mu}\int_\Omega \left[w_{x_1}^4 + w_{x_2}^4 + (1+\mu) w_{x_1}^2 w_{x_2}^2\right] dx', \\ K(t) =& \Phi_1(t) + \omega\eta\Psi(t).\end{aligned}$$

We choose $\eta \geq \omega > 0$ such that $\gamma - \eta - \omega \geq 0$ and $\nu - \omega \geq 0$.

Step 3 (properties of the functionals): Now we prove that the functionals Λ, L and K satisfy requirements (i)-(iii) in Theorem 5.2. First we rewrite the energy $\mathcal{E}$ in the form

$$\mathcal{E}(v, u, u_t) = \mathcal{E}_0(v, u, u_t) + \mathrm{Comp}\,(u),$$

where the *quadratic* energy functional $\mathcal{E}_0$ is given by (55) and $\mathrm{Comp}\,(u)$ is a functional which is continuous with respect to weak convergence. By Proposition 3.4 the functional $\mathcal{E}_0(v, u, u_t)$ provides an equivalent norm on $\widehat{\mathcal{H}}$ and therefore $\mathcal{E}(t)$ satisfies (i) by the properties of weak convergence. Now we show that Ψ is weakly continuous. We start with the third term. If $u_j \rightharpoonup u$ weakly in W, then due to Proposition 2.2 $N_0 u_j \rightharpoonup N_0 u$ weakly in $[H^{3/2}(\mathcal{O})]^3 \bigcap X$ and strongly in X. Thus, $(v_j, N_0 u^j)_{\mathcal{O}} \to (v, N_0 u)_{\mathcal{O}}$. The remaining terms are obviously weak continuous. Thus $\Lambda(t)$ satisfies (i).

Now we consider K. As above Ψ and all the terms in Φ_1 (except of Φ_0) are obviously weak continuous. The same is true for Φ_0 due to the fact that $(f_1; f_2) \mapsto f_1 \cdot f_2$ is a (strongly) continuous from $H^{1/2}(\Omega) \times H^{1/2}(\Omega)$ into $L_2(\Omega)$. Applying this property to the terms $(D_l u^i D_j w, D_m w)_\Omega$ and $(k_i w D_j w, D_m w)_\Omega$, we find that the functional K is weakly continuous and thus by the Lebesgue dominated convergence theorem the convergence of integrals in (65) holds.

As for the functional L, all its terms are obviously weakly lower semicontinuous (because of the convexity norm properties, relation (56) and Proposition 3.4), except of $E(v, v)$ and $-\varrho\|\bar{u}_t\|^2_\Omega$. Let $\{U_j\} \subset \mathcal{H}$ is bounded, $t_j \to \infty$, and $S_{t_j} U_j \rightharpoonup U$ weakly in $\mathcal{H}$. Denote by $U_j(s)$ a solution to the system under consideration with the initial data $S_{t_j} U_j$. From the energy balance equality in (25) we have that the sequence of the velocity field components v_j of $U_j(s)$ is bounded in $L_2(0, t; V)$. Therefore due to weak continuity property of the form $E(v, v)$, we obtain

$$\int_0^t e^{-\omega(t-s)} E(v(s), v(s)) ds \leq \liminf_{j \to \infty} \int_0^t e^{-\omega(t-s)} E(v_j(s), v_j(s)) ds.$$

Due to the standard trace theorem we can suppose that $\partial_t \bar{u}_j = (v_j^1|_\Omega, v_j^2|_\Omega)$ converges to $\partial_t \bar{u}$ weakly in $H^{1/2}_*(\Omega))$ (hence strongly in $L_2(\Omega)$) for almost all $t > 0$. Therefore the Lebesgue theorem yields that $\partial_t \bar{u}_j \to \partial_t \bar{u}$ strongly in $L_2(0, T; L_2(\Omega))$, so the property (iii) holds.

Thus we can apply Theorem 5.2 to prove asymptotic smoothness and complete the proof of Theorem 5.1. $\square$

Remark 5.3 We do not know whether we can relax essentially the structural force hypothesis (62) in Theorem 5.1. We use (62) to prove the boundedness of the sublevel set $\mathcal{W}_R$ only. The question on the validity of this weak form of the coercivity of the energy functional is still open in the case of general external loads.

References

[1] G. Avalos, The strong stability and instability of a fluid-structure semigroup, *Appl. Math. Optim.*, **55** (2007), 163–184.

[2] G. Avalos, R. Triggiani, The coupled PDE system arising in fluid–structure interaction. I. Explicit semigroup generator and its spectral properties, in: *Fluids and Waves*, Contemp. Math., vol. 440, AMS, Providence, RI, 2007, 15–54.

[3] G. Avalos and R. Triggiani, Semigroup well-posedness in the energy space of a parabolic hyperbolic coupled Stokes–Lamé PDE system of fluid-structure interaction, *Discr. Contin. Dyn. Sys.*, Ser.S, **2** (2009), 417–447.

[4] A.V. Babin, M.I. Vishik, *Attractors of Evolution Equations.* North-Holland, Amsterdam, 1992.

[5] J. Ball, Global attractors for semilinear wave equations, Discr. Cont. Dyn. Sys. 10 (2004), 31–52.

[6] V. Barbu, Z. Grujić, I. Lasiecka, A. Tuffaha, Existence of the energy-level weak solutions for a nonlinear fluid–structure interaction model, in: *Fluids and Waves*, Contemp. Math., vol. 440, AMS, Providence, RI, 2007, 55–82.

[7] V. Barbu, Z. Grujić, I. Lasiecka, A. Tuffaha, Smoothness of weak solutions to a nonlinear fluid–structure interaction model, *Indiana Univ. Math. J.* **57** (2008), 1173–207.

[8] A. Chambolle, B. Desjardins, M. Esteban, C. Grandmont, Existence of weak solutions for the unsteady interaction of a viscous fluid with an elastic plate. *J. Math. Fluid Mech.* **7** (2005), 368–404.

[9] I. Chueshov, A global attractor for a fluid-plate interaction model accounting only for longitudinal deformations of the plate, *Math. Meth. Appl. Sci.*, **34** (2011), 1801–1812.

[10] I. Chueshov, *Introduction to the Theory of Infinite-Dimensional Dissipative Systems.* Acta, Kharkov, 1999 (in Russian); English translation: Acta, Kharkov, 2002; *http://www.emis.de/monographs/Chueshov/*.

[11] I. Chueshov and I. Lasiecka, *Long-Time Behavior of Second Order Evolution Equations with Nonlinear Damping*, Memoirs of AMS, vol.195, no. 912, AMS, Providence, RI, 2008.

[12] I. Chueshov and I. Lasiecka, *Von Karman Evolution Equations*, Springer, New York, 2010.

[13] I. Chueshov, I. Ryzhkova, A global attractor for a fluid-plate interaction model, Preprint arXiv:1109.4324v1 (September 2011).

[14] I. Chueshov, A. Shcherbina, Semi-weak well-posedness and attractor for 2D Schrödinger-Boussinesq equations, submitted.

[15] D. Coutand, S. Shkoller, Motion of an elastic solid inside an incompressible viscous fluid, *Arch. Ration. Mech. Anal.* **176** (2005), 25–102.

[16] Q. Du, M.D. Gunzburger, L.S. Hou, J. Lee, Analysis of a linear fluid–structure interaction problem, *Discrete Contin. Dyn. Syst.* **9** (2003), 633–650.

[17] G. Duvaut, J. L. Lions, *Inequalities in Mechanics and Physics.* Springer Berlin, New York, 1976.

[18] G.Galdi, C. Simader, H. Sohr, A class of solutions to stationary Stokes and Navier-Stokes equations with boundary data in $W^{-1/q,q}$, *Math. Annalen* **331** (2005), 41–74.

[19] M. Grobbelaar-Van Dalsen, On a fluid-structure model in which the dynamics of the structure involves the shear stress due to the fluid, *J. Math. Fluid Mech.* **10** (2008), 388–401.

[20] M. Grobbelaar-Van Dalsen, A new approach to the stabilization of a fluid-structure interaction model, *Appl. Anal.* **88** (2009), 1053–1065.

[21] M. Grobbelaar-Van Dalsen, Strong stability for a fluid-structure model, *Math. Methods Appl. Sci.* **32** (2009), 1452–1466.

[22] J.K. Hale, *Asymptotic Behavior of Dissipative Systems.* Amer. Math. Soc., Providence, RI, 1988.

[23] H. Koch and I. Lasiecka. Hadamard well-posedness of weak solutions in nonlinear dynamic elasticity-full von Karman systems, in: *Prog. Nonlinear Differ. Equ. Appl. vol.50*, pp.197-216. Basel: Birkhäuser, 2002.

[24] N. Kopachevskii, Yu. Pashkova, Small oscillations of a viscous fluid in a vessel bounded by an elastic membrane, *Russian J. Math. Phys.* **5** (1998), no.4, 459–472.

[25] O. Ladyzhenskaya, *Mathematical Theory of Viscous Incompressible Flow*, GIFML, Moscow, 1961 (1st Russian edition); Nauka, Moscow, 1970 (2nd Russian edition); Gordon and Breach, New York, 1963 and 1969 (English translations of the 1st Russian edition).

[26] J. Lagnese, *Boundary Stabilization of Thin Plates*, SIAM, Philadelphia, 1989.

[27] J. Lagnese, Modeling and stabilization of nonlinear plates, *Int. Ser. Num. Math.*, 100 (1991), 247–264.

[28] I. Lasiecka, Uniform stabilizability of a full von Karman system with nonlinear boundary feedback, *SIAM J. Control Optim.*, **36** (1998), 1376–1422.

[29] I. Lasiecka, Weak, classical and intermediate solutions to full von Karman system of dynamic nonlinear elasticity, *Appl. Anal.*, **68** (1998), 121–145.

[30] I.Lasiecka, Uniform decay rates for full von Karman system of dynamic thermoelasticity with free boundary conditions and partial boundary dissipation, *Comm. Partial Diff. Eqs.*, **24** (1999), 1801–1849.

[31] J.-L. Lions, E. Magenes, *Problémes aux limites non homogénes et applications,* Vol. 1, Dunod, Paris, 1968.

[32] I. Moise, R. Rosa, and X. Wang, Attractors for non-compact semigroups via energy equations, *Nonlinearity*, **11** (1998), 1369–1393.

[33] J. Puel and M. Tucsnak, Global existence for the full von Karman system, *App. Math. Optim.* **34** (1996), 139–160.

[34] G. Raugel, Global attractors in partial differential equations, in: *Handbook of Dynamical Systems, vol. 2*, pp.885-992. Elsevier Sciences, Amsterdam, 2002.

[35] V.I.Sedenko, The uniqueness theorem of generalized solution of initial boundary value problem of nonlinear oscillations theory of shallow shells with small inertia of longitudinal displacements, *Izvestiya AN SSSR, mehanika tverdogo tela*, 6 (1991), 142–150, in Russian.

[36] V. I. Sedenko, On the uniqueness theorem for generalized solutions of initial-boundary problems for the Marguerre–Vlasov vibrations of shallow shells with clamped boundary conditions, *Appl. Math. Optim.*, **39** (1999), 309–326

[37] J. Simon, Compact sets in the space $L^p(0,T;B)$, *Annali di Matematica Pura ed Applicata, Ser.4* **148** (1987), 65–96.

[38] R. Temam, *Infinite-Dimensional Dynamical Dystems in Mechanics and Physics*, Springer, New York, 1988.

[39] R. Temam, *Navier-Stokes Equations: Theory and Numerical Analysis*, Reprint of the 1984 edition, AMS Chelsea Publishing, Providence, RI, 2001.

[40] H. Triebel, *Interpolation Theory, Functional Spaces and Differential Operators*, North Holland, Amsterdam, 1978.

[41] I. I. Vorovich, On the existence of solutions in the nonlinear theory of shells, *Izvestiya AN SSSR, Matematika* **19**(4) (1955), 173–186, in Russian.

[42] I. I. Vorovich, On some direct methods in nonlinear oscillations of shallow shells, *Izvestiya AN SSSR, Matematika* **21**(6) (1957), 142–150, in Russian.